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THE MARCH MEETING IN NEW YORK

The two hundred sixty-eighth regular meeting of the American Mathematical Society was held at Columbia University, on Friday and Saturday, March 29–30, 1929, extending through the usual morning and afternoon sessions. The attendance included the following one hundred thirty-two members of the Society:

C. R. Adams, A. A. Albert, R. L. Anderson, R. G. Archibald, H. E. Arnold, Bamforth, Bellamy, A. A. Bennett, Benton, A. C. Berry, William James Berry, William Johnston Berry, Birkhoff, Bowden, B. H. Camp, G. A. Campbell, Carlitz, Cope, L. P. Copeland, Cramlet, Dehn, Demos, Doak, Doermann, Dresden, Eisenhart, Emch, Engstrom, Feld, Fite, D. A. Flanders, Fort, R. M. Foster, Philip Franklin, Frink, Gale, Gehman, Gergen, Gill, Gronwall, C. C. Grove, G. H. Hardy, Hazeltine, Hebbert, Hildebrandt, L. S. Hill, Hille, Himwich, Hofmann, Hollcroft, Huber, R. L. Jackson, R. A. Johnson, Kaplan, Kasner, O. D. Kellogg, Kenny, Kholodovsky, Rose Klein, Kline, Knebelman, Koopman, J. P. Kormes, Mark Kormes, Kunte, Lamson, Langman, Lefschetz, Lehr, Littauer, Litzinger, Lubin, MacColl, MacInnes, Marden, Meder, Michal, E. J. Miles, H. H. Mitchell, Molina, C. N. Moore, T. W. Moore, Richard Morris, Mullins, F. H. Murray, C. A. Nelson, Newman, Ore, F. W. Owens, H. B. Owens, Parsons, Pell-Wheeler, Pfeiffer, Pooler, Post, R. G. Putnam, Raudenbush, Reddick, M. S. Rees, Rider, Ritt, Robertson, Seely, Serghiesco, Sheffer, Simons, Singer, C. E. Smith, P. A. Smith, S. E. Smith, M. H. Stone, Swartzel, Tamarkin, Evan Thomas, J. M. Thomas, J. E. Thompson, Trjitzinsky, Veblen, Wahlert, H. E. Webb, D. W. Weeks, Weida, Weinstein, Weisner, Weyl, Widder, Wiener, W. J. Willis, J. W. Young, Mabel M. Young, Margaret M. Young, Zippin.

There was no meeting of the Council or of the Board of Trustees. A telegram of greeting was received from the meeting of the Society held at the same time in Chicago, and was acknowledged appropriately.

The afternoon sessions on Friday and Saturday were devoted to a Symposium on Fourier Series. The following papers were presented: Friday, March 29, *Modern work in the theory of ordinary trigonometric series*, by Professor G. H. Hardy, with discussion by Professors Einar Hille and C. N. Moore, and Dr. T. H. Gronwall; Saturday, March 30, *Fourier series and almost periodic functions from the standpoint*

of the theory of groups, by Professor Hermann Weyl, with discussion by Professors Norbert Wiener, T. H. Hildebrandt, Philip Franklin, and M. H. Stone. Ex-President Birkhoff presided at the Symposium, relieved during the discussion on Saturday afternoon by Professor Kasner.

The morning sessions were devoted to the reading of short papers. Vice-President Young presided on Friday morning, relieved by Professor Ritt, and Professor H. H. Mitchell on Saturday morning. Titles and abstracts of the papers read at these sessions follow below. The papers of Bamforth (except his joint paper with Birkhoff), Browne, Cope (second paper), Féraud, Gronwall (second paper), Hickey, Lubin, Michal (second paper and joint paper with Peterson), Peterson, Pierpont, Rutledge, Seely, Vandiver, Walsh, Wiener were read by title. Dr. Féraud was introduced by Professor Birkhoff, Miss Hickey by Professor Michal, Mr. Hurwitz by Professor Ritt, Mr. Thielman by Professor Michal, and Mr. Whitney by Professor Pierpont.

1. Professor C. R. Adams: *Note on integro-q-difference equations.*

The equation considered is one in which the unknown function $f(x)$ is subjected simultaneously to a q-difference operator of the nth order and to an integral operator of Volterra type. Certain facts about the solutions of the equation of first order when $|q| \leq 1$ follow from the work of Picone, Popovici, Nalli and Tamarkin; these writers not unnaturally confined themselves to the real domain. The main purpose of this paper is to investigate the existence of solutions in the complex domain for the equation of order n including the case of $|q|<1$ and the essentially different case of $|q|>1$. Assuming the known functions in the equation to be analytic we find formal series solutions. When $|q|>1$ the existence of analytic solutions is shown by direct proof of convergence. In the case of $|q|<1$ results are obtained both by a generalization of Tamarkin's work and by a method of successive approximation; these methods also are employed in the real domain when the known functions are subject to less stringent hypotheses.

2. Dr. I. M. Sheffer (National Research Fellow): *The linear functional operators and equations associated with sets of polynomials.*

The writer has considered in a previous report certain algebraic properties of sets of polynomials $P: \{P_n(x)\}$, $n=0, 1, \cdots$, where the degree of $P_n(x)$ does not exceed n. Now we consider certain linear functional

operators and equations which arise when we turn to the analytic aspect of such sets. An operator L is defined as carrying the identity set I into P. We can express L as a linear differential operator of infinite order. With L we consider a second operator $\mathcal{L}$, and we investigate the characteristic numbers and characteristic functions of these operators, the problem of expanding "arbitrary" analytic functions in terms of these characteristic functions, and the question of solving the functional equations which are determined by the operators.

3. Professor C. N. Moore: *On the absolute convergence of the double Fourier series.*

In this paper it is shown that if a function of two variables which is of bounded variation also satisfies at a point a condition which is a generalization of the Lipschitz condition for functions of one variable, the double Fourier series corresponding to this function will be absolutely convergent at the same point. This theorem is a generalization to double Fourier series of a criterion found by Zygmund (Journal of the London Mathematical Society, vol. 3 (1928), p. 194) for the ordinary Fourier series.

4. Dr. F. R. Bamforth (National Research Fellow) and and Professor G. D. Birkhoff: *Divergent series and singular points of ordinary differential equations.*

Consider the system of differential equations $dx_i/dt = X_i(x_1, x_2, x_3)$ in which the X_i are analytic in x_i for $|x_i| \leq r$, $r>0$ and $X_i(0, 0, 0) = 0$, $i = 1, 2, 3$. Let the equations of variation be $dy_i/dt = \sum_{j=1}^{3} c_{ij} y_j$ where the constants c_{ij} are the values of $\partial X_i/\partial x_j$ evaluated at $x_1 = x_2 = x_3 = 0$. Let the roots m_1, m_2, and m_3 of the equation $|c_{ij} - m\delta_{ij}| = 0$ be such that m_1 and m_2 are greater than zero, m_3 is less than zero, and there exists among them no relationship of the form $m_i = p_1 m_1 + p_2 m_2 + p_3 m_3$, where the p_i are integers, positive or zero, whose sum is greater than 1. In this paper it is shown that under these hypotheses, in a certain neighborhood of the origin $x_1 = x_2 = x_3 = 0$, the system of differential equations $dx_i/dt = X_i(x_1, x_2, x_3)$ is equivalent to the system $dz_i/dt = m_i z_i$, $i = 1, 2, 3$, by means of a transformation of the type $z_i = z_i(x_1, x_2, x_3)$ where the functions $z_i(x_1, x_2, x_3)$ have the properties that they are zero for $x_1 = x_2 = x_3 = 0$, are of class C^∞, and are analytic at every point (x_1, x_2, x_3) which does not lie on a certain set of three surfaces passing through the origin.

5. Dr. F. R. Bamforth: *A boundary-value problem for a system of ordinary differential equations not linear in the parameter.*

In matrix notation in which small letters will stand for vectors of order n, capital letters for square matrices of order n, and Greek letters for numbers, the boundary-value problem studied may be written in the form $y' = y(A + B_1/(\lambda - \beta_1) + B_2/(\lambda - \beta_2))$, $My(a) + Ny(b) = 0$, where $\beta_1 \neq \beta_2$. The elements of the matrices A, B_1, B_2 are real single-valued, continuous

functions of x on the interval $a \leqq x \leqq b$, λ is a parameter, and the elements of the matrices M and N as well as β_1 and β_2 are real numbers. For boundary-value problems of the above form, definitions of adjointness and of definite self-adjointness are given. For definitely self-adjoint boundary-value problems questions relative to the existence of characteristic parameter values and the possibility of expanding an arbitrary vector in terms of the characteristic solutions are considered.

6. Dr. F. R. Bamforth: *A boundary-value problem for a system of differential equations of the second order.*

In a paper presented by the author at the Chicago meeting of the society last spring, a set of normal forms was derived for boundary-value problems connected with systems of ordinary differential equations which are linear in a parameter λ. This paper brought to light certain boundary-value problems of general type which had never been studied. The present paper is a study of one of these problems, which may be written in the form $y_1' = (a_{11}(x)+\lambda)y_1+a_{12}(x)y_2$, $y_2' = \lambda(x)y_1-(a_{11}(x)-\lambda)y_2$, $m_{11}y_1(a)+m_{12}y_2(a)+n_{11}y_1(b)+n_{12}y_2(b)=0$ $m_{21}y_1(a)+m_{22}y_2(a)+n_{21}y_1(b)+n_{22}y_2(b)=0$. Under certain hypotheses on the coefficients, it is found that there exists a denumerable infinitude of characteristic numbers. Furthermore, sufficient conditions for the simultaneous expansion of a pair of fairly arbitrary functions in terms of the characteristic solutions are given.

7. Mr. B. P. Gill: *An analog for algebraic functions of the Thue-Siegel theorem.*

In this paper the following theorem is proved. Let $\theta(t)$ be an algebraic function of t defined by an irreducible equation of degree $n \geqq 2$. Then there exist only a finite number of rational functions $x(t)/y(t)$, (x and y polynomials in t), such that, if the difference $\theta(t)-x(t)/y(t)$ be expanded about infinity in a series of descending powers of t, the initial term of the series is of degree less than $-2n^{1/2}\times$the degree of y. Applications are indicated to indeterminate algebraic equations in which the unknowns represent polynomials.

8. Mr. Solomon Hurwitz: *Algebraic first integrals of algebraic differential equations.*

It was shown by Fuchs that if an algebraic differential equation $f(x, y, y')=0$, or $y'=u(x, y)$, has a general solution $V(x, y)=c$, where V is an algebraic function of x and y, then there exists a general solution $R(x, y, u)=c$, R being a rational function of x, y, and u. In the present paper, it is shown that if an algebraic differential equation of the nth order $f(y^{(n)}, y^{(n-1)}, \cdots, y', y, x)=0$, or $y^{(n)}=u(y^{(n-1)}, \cdots, y', y, x)$, has an algebraic first integral $V(y^{(n-1)}, \cdots, y', y, x)=c$, then there exists a first integral $R(y^{(n-1)}, \cdots, y', y, x, u)=c$, where R is a rational function of the quantities in parentheses.

9. Dr. T. F. Cope (National Research Fellow): *Sufficient conditions for a weak minimum for a special case of the problem of Mayer with variable end points.*

The problem considered is that of finding conditions which suffice for a curve $y_i=y_i(x)$, $x_1\leqq x\leqq x_2$, $i=1, \cdots, n$, to minimize the first of a set of functions $f_\rho[x_1, x_2, y_j(x_1), y_j(x_2)]$, $\rho=1, \cdots, r\leqq 2n+2$; $j=1, \cdots, n$, in the class of similar curves which make $f_2, \cdots, f_r$ vanish and besides satisfy the differential equations $\theta_\alpha(x, y_j, y_j')=0$, $\alpha=1, \cdots, m<n$. First necessary conditions, namely the Euler equations and transversality conditions, were deduced by Bliss (Transactions of this Society, vol. 19 (1918), p. 305). In the author's Chicago dissertation, presented to this Society last year at the April meeting in Chicago, another necessary condition was found, which is essentially that the characteristic numbers of a boundary-value problem associated with the second variation must be greater than or equal to zero. If x_1 and x_2 are fixed, it is shown in the present paper that sufficient conditions for a weak relative minimum can be deduced from the necessary conditions mentioned above, the comparison curves being of class C'.

10. Dr. T. F. Cope: *A set of sufficient conditions for a minimum for the problem of Mayer with variable end points for specialized comparison arcs.*

The problem considered is that of the preceding paper. If the comparison curves are defined suitably, it is proved that the necessary conditions mentioned in the first paper give rise to conditions which are sufficient for a weak relative minimum even when x_1 and x_2 as well as $y_i(x_1)$, and $y_i(x_2)$ are permitted to vary. The boundary-value problem associated with the second variation in this case is of an essentially more complicated type than that of the first paper, and special methods are required to discuss it.

11. Dr. T. H. Gronwall: *Straight line geodesics in Einstein's parallelism geometry.*

The conditions that in an Einstein parallelism geometry the straight lines and the geodesics shall coincide are set up, and the resulting partial differential equations for the Z's completely integrated in the case of three dimensions.

12. Professor D. V. Widder: *On the composition of the singularities of functions defined by factorial series.*

The purpose of the present paper is to determine the form of the coefficients c_n in the series $F(s)=\sum c_n n!/[s(s+1)\cdots(s+n)]$ so that the function $F(s)$ which is defined by the series shall have singularities at most at the points $\alpha+\beta$, where the points α are the singularities of the function $f(s)=\sum a_n n!/[s(s+1)\cdots(s+n)]$, the points β those of the function

$\phi(s)=\sum b_n n!/[s(s+1)\cdots(s+n)]$. It is found under certain specified conditons that c_n may be taken as $a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$, the nth coefficient of the power series expansion of the product of the two power series $\sum a_n s^n$ and $\sum b_n s^n$. The result is obtained by expressing $f(s)$ and $\phi(s)$ as generating functions of the form $\int_0^\infty e^{-st}\alpha(t)dt$, and applying a result previously obtained and presented to the Society by the author.

13. Dr. W. J. Trjitzinsky: *On composition of singularities.*

Several theorems on composition of singularities of the type analogous to Hadamard's multiplication theorem are derived. The method is based essentially on complex integration, and on the possibility of deforming a contour unless singularities of a certain integrand tend to coincide. Let α and β be the respective singularities of two functions. Among the theorems obtained are two about a third function, in the first theorem with singularities of the form α with the exponent β, and in the second theorem with singularities of the form $(a\alpha\beta+b\alpha+c\beta+d)/(e\beta+f)$. An extension of the latter theorem, which is a generalization of Hadamard's theorem, is given for functions of two variables, similar to Haslam-Jones' extension of the ordinary multiplication theorem.

14. Professor O. D. Kellogg: *Singular manifolds among those of an analytic family.*

This paper will appear in full in an early issue of this Bulletin.

15. Dr. C. M. Cramlet (National Research Fellow): *Invariants of the n-ary q-ic differential form.*

The general differential form may be written as $a_{\rho_1\cdots\rho_q}\,dx^{\rho_1}\cdots dx^{\rho_q}$, where the $a_{r_1\cdots r_q}$ are functions of the n variables $x^1,\cdots,x^n$. Subjecting the x's to an analytic transformation and assuming the form to be invariant implies a sequence of tensors. The problem of obtaining these tensors is reduced to a problem of an affine connection Γ^α_{bc} and a tensor $a_{r_1\cdots r_q}$, where the Γ^α_{bc} are functions of position and are given explicitly in terms of the $a_{r_1\cdots r_q}$ and their derivatives. Covariant differentiation may then be defined as in affine geometry and a complete set of tensors are the $a_{r_1\cdots r_q}$ and their successsive covariant derivatives, and the curvature tensor B^r_{stu} and its covariant derivatives. The equivalence of two forms is stated in terms of a finite number of these tensors. The condition under which the present solution can be effected is that the discriminant of the form is non-vanishing. It is expected that this condition can be replaced by the condition that there exists at least one non-vanishing algebraic invariant of the tensor $a_{r_1\cdots r_q}$. Some interpretations of the significance of the tensors are given.

16. Professor A. D. Michal: *Projective functional invariants.* Preliminary communication.

This paper is concerned with the construction of functionals that are

the function-space analogs of the projective invariants of n-dimensional affinely connected manifolds. A generalization of Weyl's projective curvature tensor is obtained.

17. Professor A. D. Michal: *Tensor analysis in function space under functional transformations with differentials of the third kind.*

In an earlier paper (American Journal of Mathematics, vol. 50 (1928), pp. 473–517) the author has shown how to construct a theory of tensors in function space. In the present paper this theory is generalized and made a more harmonious whole by the employment of functional transformations with differentials of the third kind, that is, (1): $\delta z^i = {}_i'z^{(i)}\delta\bar{z}^i + {}'z_\alpha{}^i\delta\bar{z}^\alpha$. The differential functional forms are taken in the general normal form with generalizations in the differentials of their coefficients appropriate to (1).

18. Professor A. D. Michal and Mr. T. S. Peterson: *The invariant theory of functional forms under the group of linear functional transformations of the third kind.*

The invariant theory of special functional forms under the Fredholm group was initiated by A. D. Michal (American Journal of Mathematics, vol. 50 (1928)). Further contributions to this subject were given by T. S. Peterson (this Bulletin, vol. 34 (1928), p. 416). This paper is concerned with the development of an invariant theory of still more general "algebraic" functional forms in addition to the more general underlying group, that of linear functional transformations of the third kind.

19. Mr. H. P. Thielman: *On new integral addition theorems for Bessel functions and series of the hypergeometric type.*

Although Bessel functions and hypergeometric series have been studied in great detail, the literature seems to have very little to say about their integral addition theorems. This paper derives a large number of such addition theorems by employing Volterra's theories of functions of composition.

20. Miss Maude I. Hickey: *Representations of groups of linear functional transformations.*

This paper begins the systematic study of representations of groups of linear functional transformations in one function by means of linear functional transformations in a sequence of functions of several variables.

21. Mr. T. S. Peterson: *Note on the law of transformation of the nth resolvent.*

This note supplements in full the law of transformation of the nth resolvent as defined in an earlier paper (see T. S. Peterson, this Bulletin, vol. 34 (1928), p. 416).

22. Professor James Pierpont: *On the attraction of spheres in elliptic space.*

This paper has appeared in full in the May-June number of this Bulletin.

23. Professor George Rutledge: *On the relation between de la Vallée-Poussin summability and the cardinal series of interpolation.*

W. L. Ferrar (Proceedings of the Royal Society of Edinburgh, vol. 45 (1925), p. 269) has proved that, under certain very general conditions, the cardinal series of interpolation is summable by the method of de la Vallée-Poussin to the sum of the Gauss series of even order, whenever the latter series is convergent. Restricting conditions are inherent in Ferrar's method of proof, but do not in fact exist. The Gauss series of even order is identically equal, at any stage of summation, to the Lagrange polynomial of even degree, and when the latter is written in the form (previously presented by the author) $x\prod_{k=1}^{n}(1-x^2/k^2)\{f(0)/x+\sum_{r=1}^{n}\binom{2n}{n-r}\binom{2n}{n}^{-1}(-1)^r \cdot [f(r)/(x-r)+f(-r)/(x+r)]\}$, it becomes at once apparent that the sum within the braces is nothing other than the sum of $n+1$ terms of the sum (VP) of the cardinal series. Thus summability (VP) of the cardinal series and convergence of the Lagrange polynomial of even degree (and hence of the series of Gauss and Stirling, of even order) involve exactly the same limit and are equivalent without restriction.

24. Professor P. R. Rider: *On the distribution of the ratio of mean to standard deviation in small samples.*

The ratio of the mean of a sample (measured from the mean of the universe) to the standard deviation of the sample plays an important part in a number of statistical problems such as determining the probability that the mean of the sample does not deviate from the mean of the universe by more than a specified amount, comparing two mean values, and finding the sampling errors of regression coefficients. The distribution of this ratio for the case of a normal universe has been completely derived. The derivation, however, depends upon certain assumptions which do not hold for other than normal universes. The present paper derives the distribution of the ratio for certain discrete universes, with particular reference to a rectangular universe. Incidentally the distributions of other statistical parameters (mean, median, extreme average, least variate, range, and variance) for small samples are discussed.

25. Professor B. H. Camp: *Some corollaries of Bayes' theorem.*

If the incomplete Beta function is inserted as the unknown frequency function in Bayes' theorem, two results follow immediately. One is that, if m successes have occurred in n trials, the probability of a success on the

next trial is quite arbitrary, as it should be, dependent on the character of the universe from which the n trials were selected. This result compels attention to the indeterminateness of the theorem. What is frequently only a tacit assumption is now so introduced as to stand out baldly in the concluding formula. Another result is that, if, as sometimes happens, one actually does know something about the original universe from which the material under investigation was selected, this knowledge can be utilized. A further discussion is made to answer the question as to precisely what assumptions with regard to the universe the actuary and the physicist are tacitly making when they assume that the true values are near those found by experiment.

26. Professor Oystein Ore: *Remarks on the arithmetic theory of abelian fields.*

This paper contains an application of the author's arithmetic theory of Galois fields (Mathematische Annalen, vol. 100) to abelian fields. The complete system of prime-ideal types is determined, and a new proof is given for the theorem of Kronecker that all abelian fields are cyclotomic fields.

27. Dr. A. A. Albert (National Research Fellow): *The rank function of any simple algebra.*

The rank function of the general simple algebra A over F is considered. We define the central field H of A to be the field of all elements of A commutative with every element of A, and represent A as a normal algebra over H. Then the order of A over H is $q=t^2r^2$, and the rank function of A is the rank function of an associated total matric algebra A' over an extended field H', has degree tr, is irreducible in the function field $K=H(\xi_1, \xi_2, \cdots, \xi_q)$ of its coefficients, and has the symmetric group with respect to K. Then, if the order of H with respect to F is s, the order of A over F is $s^2t^2r^2$, and the rank function of A in F has degree str and is irreducible in the function field of its coefficients.

28. Dr. A. A. Albert: *The structure of algebras which are the direct products of rational generalized quaternion division algebras.*

Attempts were made, before Cecioni constructed his normal division algebras, to construct new division algebras in sixteen units by considering algebras which were the direct products of any two rational generalized quaternion division algebras. Such attempts are here shown to have been futile, as such a direct product is never a division algebra. The resulting structure of these direct products is also considered and the theorems generalized to direct products of any number of rational generalized quaternion division algebras.

29. Mr. H. T. Engstrom: *The determination of the common index divisor of an algebraic field in some general cases.*

It is shown that certain types of prime-ideal decomposition of a rational prime p in an algebraic field of nth degree determine the power of p dividing the common index divisor of the field. An explicit formula for this power of p is found. In the general case, a maximum is established for the common index divisor in terms of the degree of the field.

30. Mr. Leo Zippin: *The Janiszewski-Mullikin theorem and unbounded continuous curves.*

The author has previously analysed the relation of bounded continuous curves to the Janiszewski-Mullikin theorem. He now shows that a cyclicly connected unbounded continuous curve satisfying this theorem is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected point set. He constructs the inverse of the given curve from which it appears that the curve is a generalized cylinder (open at both ends); the curve is called a cylinder-tree, and is shown to be equivalent to a tree of elements which are its simple closed subcurves. The author discusses the case that the unbounded curve is not cyclicly connected.

31. Professor T. R. Hollcroft: *Linear systems of surfaces on a hypersurface in four dimensions.*

A linear system of surfaces on a given hypersurface in four dimensions is defined by the intersections of a linear system of hypersurfaces with the given hypersurface. The investigation includes a study of the properties of the general linear system, complete linear systems, pencils, nets and webs of a linear system, systems of surfaces on a given hypersurface cut out by its polar hypersurfaces. Linear systems of surfaces in ordinary space have been extensively studied. These systems appear as special cases of the above when the order of the given hypersurface is unity. Many of the theorems established for linear systems of three-space surfaces hold also for these more general linear systems.

32. Professor R. G. Putnam: *Note on Riesz's fourth condition for elements of accumulation.*

One of the conditions which elements of accumulation must satisfy in the abstract sets considered by F. Riesz is the following: An accumulation element A of a set E is determined by the subsets of E of which A is an accumulation element. In a Hausdorff space one of the conditions which neighborhoods must satisfy is: To every pair of distinct points x and y there correspond neighborhoods U_x and U_y which are disjoint. This note points out a relation between these two conditions.

33. Mr. Hassler Whitney: *A peculiar function.*

The function considered is continuous at points of an everywhere dense set, and discontinuous in the complementary set. At points of continuity

it has no differential coefficient, right- or left-handed, while in the complementary set the differential coefficient is definitely infinite.

34. Dr. T. C. Benton: *A definition of an unknotted simple closed curve.*

A simple closed curve may be defined as a set of points $f(t)$ where t is a parameter ranging over the interval $t_1 \leqq t \leqq t_2$ and $f(t) \neq f(t')$ unless (i) $t = t'$, or (ii) $t = t_1$, $t' = t_2$, or (iii) $t = t_2$, $t' = t_1$. If there is a function $f(t, \lambda)$ (which is uniformly continuous) such that for the range $t_1 \leqq t \leqq t_2$, $0 \leqq \lambda \leqq 1$, we have the following properties: (i) $f(t, 1) = f(t)$, where $f(t)$ is the function defining the curve above; (ii) $f(t, 0) = f_0$, a constant; (iii) $f(t', \lambda') = f(t'', \lambda'')$ if and only if (a) $\lambda' = \lambda'' = 0$, or (b) $\lambda' = \lambda'' \neq 0$ and one of the following holds: (1) $t' = t''$, (2) $t' = t_1$, $t'' = t_2$, (3) $t' = t_2$, $t'' = t_1$; then the simple closed curve $f(t)$ is an unknotted curve. It is shown that curves which satisfy this definition can be imbedded in a plane, and that those which are imbedded in a plane must be unknotted according to this definition.

35. Dr. W. L. Ayres (National Research Fellow): *Continuous curves homeomorphic with the boundary of a plane domain.*

In this note it is shown that in order that a continuous curve M be homeomorphic with a plane continuous curve which is the boundary of one of its complementary domains it is necessary and sufficient that every maximal cyclic curve of M be a simple closed curve.

36. Professor E. T. Browne: *On the signature of a quadratic form.*

In this paper a study is made of the signature of a (real) quadratic form of rank r in which $A_r \neq 0$ but which is not a regular form since we suppose that two or more consecutive terms vanish in the sequence of A's from which the signature is determined. The method employed is the rearrangement scheme of Frobenius in conjunction with Gundelfinger's rule and furnishes a very simple means of arriving at the results previously obtained in a more complicated manner by Frobenius when two consecutive A's vanish, and giving a result not given by Frobenius when three consecutive A's vanish. It is also shown how the same scheme of rearrangement may be employed in determining the signature of any real recurring form, whether regular or not.

37. Dr. T. H. Gronwall: *The number of arithmetic operations involved in the solution of a system of n linear equations.*

It is shown that, denoting by A_n, M_n, D_n the number of additions (or subtractions), multiplications, and divisions required for solving the system $\sum a_{ik}x_k = b_i$, $(i, k = 1, \cdots, n)$, by the method of substitutions, then $A_n = M_n = n(n-1)(2n+5)/6$, $D_n = n(n+1)/2$, for the most advantageous

arrangement of the method. The reduction in these numbers when the matrix (a_{ik}) is symmetric is determined, and other methods of solution are discussed.

38. Professor Edward Kasner: *Biharmonic functions and certain generalizations.*

In an earlier paper the author showed that if a function $F(x, y, x_1, y_1)$ is converted by *every* conformal substitution $x_1=\alpha(x, y)$, $y_1=\beta(x, y)$ into a harmonic function of x, y, then F must be biharmonic in the sense of Poincaré (real part of an analytic function $f(z, z_1)$ of two complex variables). The same result holds if we use merely the similitude group. If, however, we use smaller groups, for example the translation or rotation groups, larger classes of functions F are obtained, which we may then regard as generalizations of the biharmonic class. The most interesting generalization arises for the group of rigid motions. The substitutions converting an arbitrary function F into a harmonic function do not always form a group, and examples of this sort are also studied. Finally an analogous question for polygenic (non-analytic) functions of two complex variables z, z_1 is discussed: when can such a *polygenic* function, by means of a set of *analytic* substitutions $z_1=\omega(z)$, be converted into an *analytic* function of z?

39. Professor Norbert Wiener: *Group theory and harmonic analysis.*

The author shows that his methods of generalized harmonic analysis may be extended to analysis in other Eigenfunktionen than the trigonometric functions by substituting for the transformation group $f(x)\rightarrow f(x+t)$ the cyclical transformation group T^{λ}, where T^{λ} is asymptotically permutable with the transformation which makes $f(x)$ vanish when $|x|>A$.

40. Professor J. L. Walsh: *Boundary values of analytic functions and the Tchebycheff method of approximation.*

Let $f(z)$ be a function of the complex variable z analytic on the unit circle C: $|z|=1$, but with singularities interior to this circle. This paper considers approximation to the function $f(z)$ by polynomials $p_n(z)$, when the measure of the closeness of the approximation is $\epsilon_n=\max |f(z)-p_n(z)|$, z on C. It is of course true that ϵ_n cannot approach zero with $1/n$, but ϵ_n nevertheless has (for all polynomials $p_n(z)$ without restriction) a lower limit ϵ. In the simplest cases, for instance if $f(z)$ is a polynomial in $1/z$, there exists a function $p(z)$, analytic for $|z|\leq 1$, such that $\max |f(z)-p(z)|=\epsilon$, z on C, and it can be shown by the use of certain results due to Carathéodory that $\epsilon_n\rightarrow\epsilon$ implies $p_n(z)\rightarrow p(z)$ uniformly for $|z|\leq r<1$. If $p_n(z)$ is the Tchebycheff polynomial of degree n, for approximation to $f(z)$ on C, then we have $p_n(z)\rightarrow p(z)$ uniformly in a region in whose interior C lies. There are extensions to more general curves C and to more general functions $p_n(z)$ and $f(z)$.

41. Dr. Lucien Féraud: *On Birkhoff's Pfaffian systems.*

The Birkhoff Pfaffian systems of differential equations, arising from a variational principle $\delta\int_{t_0}^{t_1}[\sum_{i=1}^{2m}X_i dx_i/dt+Z]dt=0$, include those of Hamiltonian type, and are invariant in form under an arbitrary point transformation. In the present paper most of the classical properties of the Hamiltonian systems are extended to the Pfaffian systems; particular attention is paid to those properties concerning the reduction of the order of the system in case a certain number of integrals are known. A "last multiplier" of simple form is obtained. The transformation of a Lagrangian system into a Pfaffian and also the reduction of the Pfaffian form to the Hamiltonian are studied. The fundamental fact that this last reduction is possible, not only in a *formal* sense but in an *actual* sense, is established for the case of two degrees of freedom.

42. Mr. C. I. Lubin: *A note on singular points of the differential equation* $dx/[X(x,y)]=dy/[Y(x,y)]$.

The real functions $X(x, y)$, $Y(x, y)$ appearing in the differential equation $dx/X=dy/Y$ are supposed to be analytic in the neighborhood of the origin and to vanish there. The determinant made up of the coefficients of the linear terms of the power series expansions of $X(x, y)$ and $Y(x, y)$ is assumed to be different from zero and to have characteristic roots λ_1, λ_2 of opposite signs. This case has not been covered at all in the classical theory, while the recent treatment by Birkhoff by means of divergent series supposes there is no commensurability relation between λ_1 and λ_2. In this note it is proposed to obtain a normal form for the equation when $p\lambda_1+q\lambda_2=0$, where p and q are positive integers without common factor. The author proves that there exists a function $g(x, y)=y+\cdots$ involving one arbitrary constant, analytic for $x\neq 0$ and $y\neq 0$, continuous together with all its partial derivatives for $x=0$ or $y=0$, analytic in x for $y=0$ and analytic in y for $x=0$, which, for $z=x$, $w=g(x, y)$, transforms the equation into the form $dz/(\lambda_1 z)=dw/(\lambda_2 w-Az^{mp}w^{mq+1}-Bz^{2mp}w^{2mq+1})$, where A and B are constants (either or both of which may be zero) depending on the functions $X(x, y)$, $Y(x, y)$.

43. Dr. Caroline E. Seely: *Note on kernels of positive type.*

In an earlier paper (Annals of Mathematics, (2), vol. 20, p. 174) the author showed that if a kernel $K(s, t)$ is such that an infinite number of its iterated kernels are of positive type with respect to all linear combinations of its fundamental functions $\Phi_i(s)$, $\Psi_i(s)$, then $\int_a^b\Phi_i(s)\Phi_j(s)ds=0$, $\int_a^b\Psi_i(s)\Psi_j(s)ds=0$, $\lambda_i\neq\lambda_j$. In the present note it is shown by an example that this condition is not necessary in order that $K(s, t)$ itself be of positive type with respect to every function developable in a Fourier series.

44. Professor H. S. Vandiver: *On the first case of Fermat's last theorem.*

In this paper the author examines the equation $x^l+y^l+z^l=0$, where

x, y, and z are rational integers prime to the odd prime l and none zero. The special case $x \equiv y \pmod{l}$ is discussed, and among other results the following theorem is obtained: If the above equation is satisfied under the conditions mentioned, and $x \equiv y \pmod{l}$, then there exists no prime integer in the set $1+2l, 1+4l, \cdots, 1+(l-1)l$. This paper will appear in the Annals of Mathematics.

45. Professor H. S. Vandiver: *A theorem of Kummer's concerning the second factor of the class number of a cyclotomic field.*

This paper has appeared in full in the May-June number of this Bulletin.

46. Professor H. S. Vandiver: *An algorithm for transforming the Kummer criteria in connection with Fermat's last theorem.*

In this paper the author applies the theory of power characters in the field $k(\zeta)$, $\zeta = e^{2i\pi/l}$, l an odd prime, to the equation $x^l+y^l+z^l=0$, where x, y, and z are rational integers prime to l and none zero. By means of this method various combinations of the well known Kummer criteria for the solution of this equation are obtained, including new derivations of the congruences $2^{l-1} \equiv 3^{l-1} \equiv 1 \pmod{l^2}$. This paper will appear in the Annals of Mathematics.

ARNOLD DRESDEN,
Associate Secretary

THE MARCH MEETING IN CHICAGO

The two hundred sixty-ninth regular meeting of the Society was held at the University of Chicago on Friday and Saturday, March 29–30, 1929. About one hundred fifty persons attended the meeting, among whom were the following one hundred fourteen members of the Society, the largest number of members ever attending a meeting in Chicago:

R. W. Babcock, Bardell, Barnard, Barnett, Bartky, Beenken, Bibb, G. A. Bliss, Boyce, Brand, Brixey, O. E. Brown, L. H. Bunyan, Butler, Cairns, G. N. Carmichael, R. D. Carmichael, E. H. Carus, Chittenden, Coble, H. H. Conwell, Copp, H. V. Craig, Curtiss, Dancer, H. A. Davis, Dickson, P. D. Edwards, Eiesland, Escott, H. P. Evans, Everett, Feltges, Georges, Germond, Gilmore, J. W. Glover, Gouwens, L. M. Graves, Griffiths, V. G. Grove, Hartung, Hawkins, E. R. Hedrick, Holgate, Louis Ingold, Ingraham, Dunham Jackson, Jonah, B. W. Jones, Kern, Ketchum, Knight, E. P. Lane, Langer, LaPaz, Larsen, Latimer, Longley, McCoy, McNair, MacDuffee, MacMillan, March, T. E. Mason, Merriman, E. H. Moore, Max Morris, D. C. Morrow, E. J. Moulton, Musselman, Oakley, Olson, Palmer, Parkinson, Pepper, Powell, Price, Rainich, Rashevsky, R. G. D. Richardson, H. L. Rietz, Ross, Roth, Rowland, Sanger, Schottenfels, Sheets, Shohat, R. C. Shook, W. G. Simon, Skinner, Slaught, E. R. Smith, Sokolnikoff, Speeker, Stafford, Stouffer, M. E. Taylor, Teach, J. S. Turner, Van Vleck, Vass, Wahlin, Wall, L. E. Ward, Warren Weaver, Weiss, Wiggin, R. L. Wilder, Wiley, K. P. Williams, F. E. Wood, F. L. Wren.

On Friday afternoon Professor E. B. Van Vleck gave a symposium address on *The location of roots of polynomials and entire functions*. A rising vote of thanks was given the speaker.

On Friday evening members and their guests attended a dinner in the Del Prado Hotel. Professor Holgate was toastmaster. He called upon Professor R. G. D. Richardson who spoke of the aspirations and future plans of the Society. Professor G. A. Bliss then spoke in appreciation of the scientific work and influence of Professor Van Vleck, and the toastmaster read a letter from Professor Birkhoff on the same subject. Professor Holgate then called on Professor Van

Vleck, and afterwards on Professor E. H. Moore. The last speaker was President Hedrick, who spoke of Professor Van Vleck's work and its lessons for the future. The dinner was attended by ninety-four persons. There was a telegraphic exchange of greetings between the groups meeting in Chicago and in New York.

Sessions of the Council were held both before and after the dinner, the President and Secretary being in attendance.

It was announced that the following twenty-eight persons were elected to membership:

Professor Edwin Brown Allen, Rensselaer Polytechnic Institute;
Miss Jean C. Hempstead, Drexel Institute;
Mr. Everett H. Johnson, Lehigh University;
Mr. Paul Allen Knedler, University of Pennsylvania;
Mr. Clarence Anding Lovell, Drexel Institute;
Mrs. Elizabeth Elliott Nixon (Mrs. J. C.), University of Alabama;
Miss Irene Price, Indiana University;
Mr. John Albert Roulton, engineer, Rainey-Wood Coke Company, Swedeland, Pa.;
Mr. Ronald G. Smith, Kansas University;
Professor Guy Stevenson, University of Louisville;
Mr. Alexander Tartler, Drexel Institute;
Mr. Charles Henry Vehse, Brown University;
Mr. Ralph Oaks Virts, Indiana University;
Professor Horatio Burt Williams, Columbia University;

Nominees of the Mathematical Association of America:

Professor I. F. Neff, Drake University, Des Moines, Iowa;
Mr. J. C. Nixon, University of Alabama, Tuscaloosa, Ala.;
Mr. R. E. Peterson, Pennsylvania State College, State College, Pa.;

Nominees of the University of Pennsylvania:

Mr. V. W. Adkisson; Mr. Leonard Carlitz; Mr. Alexander G. Makarov;

Nominees of the Metropolitan Life Insurance Company:

Mr. George V. Brady; Mr. J. Christman; Mr. Reinhard A. Hohaus;

Nominees of Members of the Department of Mathematics, University of Wisconsin:

Mr. R. Erickson; Mr. H. H. Germond; Mr. E. Hawkins; Mr. H. D. Larsen; Mr. J. S. McNair.

In pursuance of the action taken at the Annual Meeting of 1928, the San Francisco Section has been abandoned. Professor B. A. Bernstein was appointed as Associate Secre-

tary and Professors M. W. Haskell (chairman), E. T. Bell, H. F. Blichfeldt, and R. M. Winger on the Committee on Program to work with him, in organizing the future meetings of the Society on the Pacific slope.

It was decided that the meetings of the Southwestern Section be stopped and that the following working plan govern the allocation of meetings in the Middle West:

1. In case the Western Meeting of the Society in connection with the Association for the Advancement of Science in December is held in one of the two regions southeast of Chicago or southwest of Chicago, a meeting of the Society on the Saturday after Thanksgiving of the same year shall be held in the other region.

2. In case the above mentioned December meeting of the Society is not held in either region during a given year, but the Summer Meeting of the Society that year is held in one of the two regions, a meeting of the Society shall be held that year on the Saturday after Thanksgiving in the other region.

3. In case neither the December Meeting nor the Summer Meeting is held in one of the two regions during a given year, the Council shall authorize a meeting to be held on the Saturday after Thanksgiving that year in either the region southeast or the region southwest of Chicago, somewhat in rotation.

There are now no sections remaining and all meetings held will be meetings of the Society; hereafter all meetings are to be numbered seriatim and subsidiary numbering abandoned.

The 1929 Thanksgiving meeting of the Western group is to be held at Ann Arbor. Invitations from President Vinson of Western Reserve University and President Howe of Case School of Applied Science to hold a meeting in Cleveland in connection with the 1930–31 meeting of the American Association for the Advancement of Science were accepted.

The following appointments were made: as a committee to nominate officers and members of the Council: Professors O. D. Kellogg (chairman), C. R. Adams, R. D. Carmichael, L. E. Dickson, and Tomlinson Fort; as a committee for the

meeting in Ann Arbor: Professors J. W. Glover (chairman), J. W. Bradshaw, T. H. Hildebrandt, M. H. Ingraham, and W. G. Simon; as a committee for the meeting in Cleveland: Dean T. M. Focke (chairman), Professors W. D. Cairns, W. B. Ford, M. H. Ingraham, and W. G. Simon; to represent the Society on the Council of the American Association for the Advancement of Science for 1929: Professors Louis Ingold and M. H. Ingraham; to represent the Society on the American Year Book: Professor Tomlinson Fort.

Mr. Charles A. Cole has made a gift to the Society, in memory of his father, which more than doubles the Frank Nelson Cole Prize Fund. In thanking him, the Council expressed its conviction that this generous action will prove an important factor in the furtherance of those high ideals of the Society toward the establishing of which the father so notably contributed.

A memorandum of the financial needs of the Society as prepared at the request of the General Education Board was presented to the Council.

Minor changes in the by-laws were recommended to the Society for action at the Summer Meeting.

In view of the tremendous demands for space in the Bulletin, the editors of the Bulletin and Transactions were requested to study the problem of journal publication for the Society. The appointment also was authorized of a committee to investigate the demand and the probable financial support for a journal of applied mathematics.

The papers whose abstracts appear below were read in three sections: (1) Analysis—Friday morning; (2) Geometry and Point Sets—Friday morning; (3) General—Saturday morning. President Hedrick, Professors Van Vleck and Bliss presided at the first; Professors Lane and Ingold at the second; President Hedrick and Professor Rietz at the third. Papers 1–14 were read in the first section; papers 15–23 in the second; and papers 24–39 in the third. Papers 10, 11, 13, 14, 23, 35–38 were read by title.

1. Dr. H. S. Wall: *On the Padé table of approximants associated with a positive definite power series.*

This paper studies the diagonal files in the Padé table for series in which the Stieltjes inequalities $A_n>0$, $B_n>0$, are replaced by the single condition $A_n>0$. The results are somewhat analogous to those obtained by the writer for the less general Stieltjes series (Transactions of this Society, vol. 31 (1929), pp. 91–116). Let S_0, S_1, S_2, $\cdots$ denote the principal diagonal file and the successive parallel files to the right of it respectively, in the Padé table. Then all the files (1) S_0, S_2, S_4, $\cdots$ converge to limits which are analytic over any region not containing a part of the real axis. The files S_1, S_3, $\cdots$ need not converge. There are just three cases. Case i, all the files (1) have a common limit; Case ii, the files S_0, S_2, $\cdots$, S_{2k} have a common limit while no two of the subsequent files have equal limits; Case iii, no two of the files (1) have equal limits. Similar results are obtained for the files below the principal diagonal. In cases ii, iii the limits are all meromorphic functions of $1/z$.

2. Dr. H. S. Wall: *On the expansion of an integral of Stieltjes.*

This paper studies the formal expansion in descending powers of z of the integral $\int_0^\infty [(1-\lambda)/(e^{(1-\lambda)u}-\lambda)]e^{-zu}du$, a continued fraction development of which was given by Stieltjes, *Recherches sur les fractions continues* (Oeuvres, vol. 2, p. 547). The $(n+1)$st coefficient in this expansion is a reciprocal polynomial of degree $n-1$ in λ: $c_n(\lambda)=\sum_{i=0}^{n-1}a_{n,i}\lambda^i$, having only real, negative, simple roots, the roots of c_n being separated by those of c_{n-1}. Since $c_n(1)=n!$, the coefficients constitute a partition of $n!$, the number of elements in the symmetric group $G_{n!}$. Let t take k into i_k. Let $K_{n,p}$ denote the set of substitutions t in which exactly p inequalities $i_{k+1}>i_k$ obtain. Then $K_{n,p}$ contains $a_{n,p}$ elements and $T^{-1}K_{n,p}T=K_{n,p}$, where $T=(1,\ n)(2,\ n-1)\ (3,\ n-2)\cdots$. The sets $K_{n,p}$ and $K_{n,n-p-1}$ contain the same number of conjugate pairs e_p, $\bar{e}_p=T^{-1}e_pT$ and e_{n-p-1}, $\bar{e}_{n-p-1}=T^{-1}e_{n-p-1}T$ respectively, and also equal numbers of elements g_p, g_{n-p-1}, respectively, self-conjugate with respect to T. The set of all elements g forms a group of order $k!\,2^k$ when $n=2k$ or $2k+1$. Certain multiplication laws hold among the four elements e_p, $Te_p=e_{n-p-1}$, $\bar{e}_p$, and $T\bar{e}_p=\bar{e}_{n-p-1}$ and also among the elements $g_p, Tg_p=g_{n-p-1}$ and $g_p', Tg_p'=g_{n-p-1}'$.

3. Professor J. A. Shohat: *On a certain formula of mechanical quadratures with non-equidistant ordinates.*

The author investigates the following formula of mechanical quadratures:

$$\text{(A)}\qquad \int_{-n}^{n} u_x dx = a(u_{-n}+u_n)+\sum_{i=1}^{n}a_i(u_{-n_i}+u_{n_i}),\qquad (u_x\equiv u(x));$$

$(0<n_i<n)$, which has been discussed for $s=2$, 3 by G. F. Hardy, A. King and J. Glover. The existence of (A) is established for any positive integer s,

also its convergence for $s \to \infty$; the properties of the $n_i(s)$ and $a_i(s,a)$, also their numerical values for $s=3$, 4 are given. This paper will appear in full in the Transactions of this Society.

4. Professor G. A. Bliss: *An integral inequality.*

This paper contains a proof for $n>m>1$ of the inequality

$$\int_0^\infty y^n dx/x^{n-r} \leqq \left(\int_0^\infty f^m dx\right)^{n/m} \qquad (r=n/m-1),$$

when the integral in the second member exists in the sense of Lebesgue and

$$y=\int_0^x f\,dx, \quad k=(1/(n-r-1))\,[r\Gamma(n/r)/\Gamma(1/r)\Gamma(n-1/r)]^r.$$

The functions f for which the equality holds are specified. The method used constitutes a very interesting application of the calculus of variations to a case which is somewhat irregular. The inequality was suggested by Professor G. H. Hardy, and the paper is in answer to a query from him as to whether a proof by calculus of variations methods is possible. No other proof has as yet been given so far as is known to the writer.

5. Dr. Lincoln La Paz: *The Hamiltonian function and an inverse problem of the calculus of variations.*

The problem solved is the following: Given a function $H(x, y_1, \cdots, y_n, u_1, \cdots, u_n)$ arbitrary except for suitable continuity properties, under what conditions is this function the Hamiltonian function of a problem of the calculus of variations in $(x, y_1, \cdots, y_n)$-space? Necessary and sufficient conditions are found to be that H be of class C''' in a region S of $(x, y_1, \cdots, y_n, u_1, \cdots, u_n)$ values and that the determinant $|\partial^2 H/\partial u_i \partial u_m|$ be different from zero in S. The method used leads to an explicit expression for the integrand function of the variation problem associated with an admissible function H. As a corollary we obtain the conditions under which a prescribed first-order partial differential equation of the form $F(x, y_1, \cdots, y_n, W_x, W_{y_1}, \cdots, W_{y_n})=0$ can be the Hamilton-Jacobi partial differential equation of a problem of the calculus of variations in $(x, y_1, \cdots, y_n)$-space. Paulus has devoted a paper (Mathematische Zeitschrift, vol. 25 (1926), pp. 348–361) to the solution of this last problem in the special case $n=1$. His argument is based on properties of the Hilbert invariant integral rather than of the Hamiltonian function.

6. Dr. P. W. Ketchum: *Solution of partial differential equations by hypervariables.*

Consider the differential operator $P(p_1, \cdots, p_n)$, where $p_i=\partial/\partial w_i$, and P is a homogeneous polynomial of degree r, with constant coefficients. The necessary and sufficient conditions that the real part of all analytic functions $F(w)$, of the hypervariable $w=\sum w_i e_i$, be a general solution of the equation $P \cdot f(w_1, \cdots, w_n)=0$ are that $P(e_1, \cdots, e_n)=0$, and that the real part of αw^s contain a sufficient number of arbitrary constants. The number of such constants depends on n and on r, and, if n is greater than 2, the num-

ber increases indefinitely with t. It is thus seen why it is necessary to use an infinite hypervariable for Laplace's equation. By application of the above theorem, hypervariables were found which furnish general solutions of the wave equation and of Airy's equation.

7. Professor L. E. Ward: *On the region of convergence of a certain type of series.*

The series considered is that consisting of the characteristic functions of the irregular boundary value problem $d^2u/dx^2+\rho^2u=0$, $u(\pi)=u(-\omega_2\pi)=u(-\omega_2\pi)$. The author has shown (Transactions of this Society, Oct., 1927, pp. 741–745) that if such a series converges uniformly in any segment of the interval $(0, \pi)$, it will converge uniformly in and on any equilateral triangle T centered at $x=0$ and having one vertex at an interior point of the segment of assumed uniform convergence; and also that any function analytic in a region containing $x=0$ as an interior point can be expanded in such a series uniformly convergent in and on a certain triangle T. By examining the magnitude of the coefficients of the formal expansion of a function, the author shows in the present paper that if the function is analytic in a region of which $x=0$ is an interior point and also satisfies certain continuity conditions, the triangle of uniform convergence can be taken as large as one likes except that it shall not contain within or upon itself either a singularity of the function or the point π.

8. Professor P. D. Edwards: *Functions possessing an addition theorem from the standpoint of a functional equation given by Abel.*

The usual procedure in the study of functions having addition theorems has been to define a function $\theta(x)$ and then to determine whether or not there exists a function f such that $\theta(x+y)=f(\theta(x), \theta(y))$. This paper inverts the problem and looks at it from the standpoint of the function f. A necessary and sufficient condition that f represent an addition theorem is that it satisfy a functional equation given by Abel. This functional equation is used to find explicit formulas for f in the case where f is rational and in certain irrational cases. The function θ corresponding to a given f is found by the aid of a theorem due to Weierstrass, and thus the addition theorems of most of the well known functions are obtained.

9. Professor Louis Brand: *On the transformation of surface integrals to circuit integrals.*

The parent theorem of all transformation of this class is

$$\int n \times F d\sigma = \int tF ds, \tag{1}$$

where F is a scalar, vector or tensor function of position and ∇F is either a surface or space gradient. When F is a tensor function, many other theorems follow from this by equating invariants of both members which are distributive with respect to addition. One such theorem is

$$\int \{\nabla F \cdot n^{(2)} - n \operatorname{div} F\} d\sigma = \int t \times F ds, \tag{2}$$

where the index 2 means that the operation $\cdot n$ applies to the second vector of the tensor preceding and div F denotes a surface divergence if ∇F is a surface gradient. Now if f is a second function of the same generality as F and we put $F = nf$ in (2), we obtain the integral transformation

$$\int (\nabla f + J n f)\, d\sigma = \int t \times n f\, ds, \tag{3}$$

J being the mean curvature. This transformation is coextensive with (1); this may be shown by putting $f = nF$ in (3) and equating cross invariants. From (3) in turn other useful theorems may be deduced by the process of equating distributive invariants. One of these theorems finds an important application in the problem of minimizing a surface integral limited by a fixed closed curve.

10. Mr. W. T. Reid: *A boundary-value problem for a countable system of ordinary linear differential equations.*

The system considered is (1) $y'(x, \mu) = A(x, \mu)y(x, \mu)$ where $y(x, \mu)$ is an absolutely continuous vector in Hilbert space on X: $0 \leqq x \leqq 1$ and for values of the parameter μ on L: $L_1 < \mu < L_2$ the matrix A is summable on X and limited by the summable function $\phi(x)$ uniformly for μ on L; furthermore, for x almost everywhere on X and μ on L, $A(x; \mu)$ is a *vollstetig* matrix, as defined by Hilbert. This system with the following boundary conditions (2) $[E+G(\mu)]y(0; \mu)+[E+H(\mu)]y(1; \mu) = 0$, where E is the unit matrix, G and H are "vollstetig" matrices and $E+G$ and $E+H$ have unique reciprocals for each value of μ, has many of the properties of a finite system. When it is incompatible we may define uniquely a Green's matrix. It is proved that if for $\mu = \mu'$, system (1), (2) is compatible there exists a reduced system (1'), (2) which is incompatible, where (1'), $y'(x) = [A(x; \mu') + P(x)]y(x)$ and P is a matrix of non-negative continuous functions which is limited on X by an arbitrarily assigned ϵ.

11. Mr. W. T. Reid: *Boundary conditions for a countable system of linear differential equations.*

Two infinite matrices M and N that are limited, as defined by Hilbert, are said to belong to class $D_1[D_2]$ if there exists an infinite matrix (M', N') $[\binom{M'}{N'}]$ composed of columns [rows] of M and N which possesses a unique reciprocal, and furthermore there exist an infinite number of columns [rows] of M and N which do not belong to (M', N') $[\binom{M'}{N'}]$. We consider the infinite system $y'(x) = A(x)y(x)$, where $y(x)$ is an absolutely continuous vector in Hilbert space and the matrix $A(x)$ is summable and limited by a summable function $\phi(x)$ on X: $0 \leqq x \leqq 1$, with boundary conditions $My(0)+Ny(1) = 0$, where the matrices M and N are of class D_1. It is shown that for the adjoint system $z'(x) = -z(x)A(x)$ it is possible to determine boundary conditions $z(0)P+z(1)Q = 0$, where P and Q are matrices of class D_2 and $MP-NQ = 0$.

Such a definition of adjoint boundary conditions reduces for a finite system to that given by Bounitzky.

12. Professor H. W. March: *The field of a magnetic dipole in the presence of a conducting sphere.*

A plane coil through which an alternating current is flowing may be regarded as an alternating magnetic dipole at distances from the coil which are great in comparison with its dimensions. In this paper the general problem of the field of such a dipole in the presence of a conducting sphere is solved. From the general formulas, approximate expressions are obtained for the field components for the case in which the radius of the sphere is small in comparison with the wave length.

13. Dr. J. M. Earl: *The degree of convergence of Hermite and Laguerre series.*

In a previous paper dealing primarily with a more general problem (Bulletin of this Society, vol. 34, p. 416), the writer has discussed incidentally the degree of convergence of the Hermite development of a given function, the treatment being based directly on the fact that the partial sum $S_n(x)$ of order n is a polynomial of the nth degree which makes the integral of $e^{-x^2}[f(x)-S_n(x)]^2$ from $-\infty$ to $+\infty$ a minimum. By use of an asymptotic expression for the Hermite polynomial of degree n, it is now shown that on any finite interval, $S_n(x)$ does not exceed a constant multiple of $M \log n$, where M is the least upper bound of $|f(x)|e^{-hx^2}$ for all real values of x, and h is any given number less than $1/2$. This leads to improved results with regard to the degree of convergence of the Hermite series. Corresponding treatment is applicable to the Laguerre development on the simply infinite interval.

14. Mr. E. L. Mickelson: *On the approximate representation of a function of two variables.*

The first four sections of this paper extend known theorems on finite trigonometric and polynomial sums approximating a function of one variable which possesses a modulus of continuity to the case of a function of two variables which is similarly restricted. The first two consist of theorems on the existence of trigonometric and polynomial approximating double sums, the third treats double sums which are trigonometric in one variable and polynomial in the other, and the fourth applies the foregoing to discussion of the degree of convergence of certain double series—Fourier, Legendre, and a combination of these. In general terms, the fifth section proves that if a function possesses rth order directional derivatives of a certain type which have a modulus of continuity, $\omega(d)$, on the surface of a unit sphere, the sum of the first $m+1$ terms of Laplace's series for the function approximates it to the order of $\omega(2\pi/m)m^{1/2}(1/m)^r$, for all finite, positive, integral values of r, including $r=0$.

15. Professor J. R. Musselman: *A classification of planar six points.*

The author has made a classification of six points in the plane according to equalities existing between their irrational invariants. These invariants were developed by Coble, Transactions of this Society, vol. 16 (1915). In the present paper the author has divided six points into a certain number of types and developed canonical forms for them.

16. Dr. H. A. Davis: *Non-involutorial transformations belonging to a non-special linear complex.*

A brief synthetic discussion of non-involutorial transformations belonging to a non-special linear complex has been given by D. Montesano. The present paper extends the synthetic discussion and investigates the problem analytically. The equations of the general transformation are obtained and discussed.

17. Dr. H. A. Davis: *Non-involutorial transformations belonging to a special linear complex.*

The equations of the general non-involutorial transformation which belong to a special linear complex are here obtained and discussed. The analytic work reveals an error in the synthetic discussion of the problem by M. Pieri (Circolo Matematico di Palermo, vol. 6). This error causes a considerable part of Pieri's paper to be incorrect, especially the F- and P-systems of the transformation. In the present paper the equations of the components of the latter are obtained and discussed. It is also shown that the general transformation can be built up from the $T_{3\to3}$ in a plane through the directrix of the special linear complex. The study of this plane $T_{2\to2}$ reveals some interesting facts concerning the F- and P-systems of the space transformation.

18. Professor E. P. Lane: *On the contact of a quartic surface with an analytic surface.*

Since there are $(k+1)(k+2)/2$ conditions for an algebraic surface to have contact of order k at a point of an analytic surface, and since an algebraic surface of order h depends on $h(h^2+6h+11)/6$ parameters, there is a limitation on the order of contact possible for a given algebraic surface. A quartic surface depends on 34 parameters, so that there is a six-parameter family of quartics having contact of the sixth order at a point of an analytic surface. It is the primary purpose of this paper to initiate an investigation of the contact of a general quartic surface with an analytic surface. The cases in which the quartics have contact of the fourth, fifth, and sixth orders receive particular attention. In each case the possibilities as to composite quartic surfaces are considered, and emphasis is laid on the curve of intersection of a non-composite quartic and the tangent plane at the point of contact.

19. Professor May M. Beenken: *Surfaces in five-dimensional space.*

The method of Wilczynski is extended here to a systematic study of the projective differential geometry of surfaces in five-dimensional space S_5. A general surface in S_5 is determined, except for a projective transformation by a completely integrable system of four linear homogeneous partial differential equations of the third order. A local pyramid of reference is set up, and equations of linear osculants and of other linear spaces related to the surface are obtained in the local coordinate system. Quasi-asymptotic and principal curves, invented by the Italians as analogs of the asymptotic and conjugate curves on surfaces in ordinary space, are treated by a new method. Special surfaces in S_5, which sustain a conjugate net or one family of asymptotic curves, are investigated by means of other completely integrable systems of differential equations.

20. Professor John Eiesland: *On a class of ruled* $(n-1)$*-spreads in* S_n*, the* V_4^4 *in* S_5.

The locus of all the ∞^{n-2} lines that are incident with n given generic $(n-2)$-flats in S_n is a ruled $(n-1)$-spread. This spread has been studied by Segre for the case $n=4$. In a recent article in the Bulletin of this Society, vol. 34 (1928), B. C. Wong has constructed the generalized spread V_{n-1}^{n-1} synthetically, but owing to the limitation of this method in the case of n-dimensional geometry, only a few of the more obvious properties of the spread have been obtained. In the present paper a strictly analytic method has been adopted. The equation of the general V_{n-1}^{n-1} in S_n has been obtained and a more detailed discussion of the geometric properties of the V_4^4 in S_5 is given. The introduction of the Grassmannian coordinates of a 3-flat in S_5 is a special feature of this part of the work.

21. Professor R. L. Wilder: *On the imbedding of subsets of a metric space in Jordan continua.*

Let R be a metric space, and M a subset of R. Several conditions which are both necessary and sufficient in order that M should be a subset of a Jordan continuum ("continuous curve") J of R are obtained. Two of these follow: (1) If M is compact and closed, it is necessary and sufficient that M should be arcwise connected through R as well as arcwise connected im kleinen through R. In this case, J can be so chosen that $J-M$ is the sum of a denumerable set of non-overlapping open arcs $\{L_i\}$ such that $\lim_{i\to\infty} d(L_i)=0$, and such that if M has positive (Menger-Urysohn) dimension, J is of the same dimension. (2) If M is compact, it is necessary and sufficient that there should exist in R a denumerable set of points $\{P_i\}$ which is dense on M, and such that the set of points $\{P_i\}$ is arcwise connected through R as well as uniformly arcwise connected im kleinen through R. In this case, if the dimension of M is m, and the dimension of the outer shell of M is s, then J may be so chosen that its dimension is the greater

(or common) value of m and s, provided, of course, that either m or s is positive.

22. Professor R. L. Wilder: *On the relation of certain sets to their complements in euclidean spaces of three and higher dimensions.*

This paper is an attempt to establish some fundamental relations between certain basic sets and their complements in euclidean spaces of three and higher dimensions. In particular, a result of R. L. Moore (Proceedings of the National Academy, vol. 4 (1918), p. 369) to the effect that each of the domains complementary to a simple closed curve in the plane is uniformly connected im kleinen is extended to higher spaces as follows: In E_n (euclidean space of n dimensions) each of the domains complementary to a set M which is homeomorphic with an $(n-1)$-sphere, or, indeed, an $(n-1)$-manifold, is uniformly connected im kleinen. Furthermore, every point of M is regularly accessible by arcs from each of the complementary domains of M. A generalization of the theorem that every point of an arc in E_3 is accessible from its complement (see letter from C. Kuratowski to R. L. Moore, this Bulletin, vol. 31 (1925), p. 32) is obtained as follows: Every point of an i-cell $(0 \leqq i \leqq n-1)$ immersed in E_n is accessible by arcs from the complement of the cell.

23. Professor Arnold Emch: *On the geometry of symmetric functions.*

The geometric properties of symmetric substitution groups were discussed in a paper which appeared in the American Journal of Mathematics, vol. 45 (1923). In the present paper the author continues the study of symmetric functions from a geometric standpoint and investigates the properties of a number of algebraic curves which are invariant under finite collineation-groups, and also Cremona groups. The 8 cycles of order 3 on an elliptic cubic, for example, give rise to 4 elliptic class-sextics of order 12, which bear interesting relations to the cubic. Other examples for groups G_{12}, G_{24}, G_{48}, G_{108} are given.

24. Mr. O. E. Brown: *The equivalence of triples of bilinear forms.*

In the sense of Weierstrass, two triples of bilinear forms are equivalent if and only if their matrices are equivalent. We restrict the triples to those whose leading matrix is non-singular and which, thus, may be taken as the identity. Let the pairs (I, b) and (I, c) have the classical canonical forms (I, λ) and (I, μ). The triple (I, b, c) is equivalent to two classes of triples (I, λ, c_1) and (I, b_1, μ) called the first and second canonical neighborhoods. A transformation reducing a matrix of the first canonical neighborhood to the second is called a transposition. The set of all transpositions is the same for all triples equivalent to a given one.

25. Professor M. H. Ingraham: *An application of a theorem on matrices to the theory of equations.*

This paper calls attention to the use of the fact that if r_i, $i=1, \cdots, n$, are the roots of the characteristic equation of a matrix m, the roots of $|g(m, x)|=0$ can be divided into sets s_{ik} associated with the r_i in such a way that $g(r_i, s_{ik})=0(i, k)$, in finding an equation whose roots are connected by a polynomial equation with the roots of a given equation. All functions considered are polynomials. This is a generalization of results of A. Chatelet (Annales de l'Ecole Normale, (3), vol. 28 (1911), p. 124).

26. Professor L. E. Dickson: *All integral solutions of certain diophantine equations.*

Let $f(x, y)$ be a binary quadratic form whose discriminant d is not a square. Given one rational solution of $f=ez^2$, we can express all rational solutions as the products of ρ by certain quadratic functions X, Y, Z of parameters u and v, which are relatively prime integers. Express ρ as an irreducible fraction N/D. Then our solution is integral only when D divides X, Y, Z. It is proved that D then divides de. Hence all integral solutions are given by as many formulas (each involving only integral parameters) as there are positive divisors of de. For certain values of m and n all integral solutions are obtained of the problem to find two sets of n integers having the same sum, same sum of squares, $\cdots$, and same sum of mth powers.

27. Professor L. E. Dickson: *Binary quadratic forms with a single class in each genus.*

Let the form have a negative discriminant $-\Delta$. For Δ odd, $\Delta=4n-1$. For $ac=n$, $ax^2 \pm xy+cy^2$ are of disciminant $-\Delta$ and are both reduced for certain a, c, unless n is a prime or square of a prime. By using a form with the term $\pm 3xy$, when n is a prime, we conclude that $n+2$ must be a prime, a square of a prime, a triple of a prime, or 27 or 81. By means of such results we readily obtain the few answers as far as 23,000. The method is not quite so simple when Δ is even.

28. Professor C. C. MacDuffee: *On the independence of the first and second matrices of an algebra.*

This paper has appeared in full in the May-June issue of this Bulletin.

29. Professor G. A. Bliss: *Expansions for implicit functions near a very general type of singularity.*

In his Colloquium Lectures of 1909 the writer studied for real variables the inverse of a transformation of the form (1) $x=u+\phi_2+\cdots$, $y=\psi_2+\cdots$, where ϕ_2, ψ_2 are homogeneous of the second degree in u, v and the dots indicate terms of higher orders. The functional determinant of the second members of these equations with repect to u, v will be repre-

sented by $D(u, v)$. In a recent letter Professor J. L. Synge of the University of Dublin writes that W. R. Hamilton in his first published paper was led in 1828 to this same transformation in his effort to compare the intensity of light near a caustic surface in an optical field with the intensities in other portions of the field. By a lengthy and complicated argument, Hamilton finds that the solutions u, v of equations (1) can be expressed as power series in x, $[y-\phi(x)]^{1/2}$, where $y=\phi(x)$ is the locus of the points in the xy-plane corresponding to the zeros of $D(u, v)$. In this paper the writer shows that similar ones can be deduced for singular points of implicit functions defined by the much more general equations of the form $f_r(x_1, \cdots, x_n; y_1, \cdots, y_n) = 0$, $(r=1, \cdots, n)$. The singular points near which such expansions hold are of a very general type since their definition imposes only one inequality on the coefficients of the expansions of the functions f_r besides the vanishing of the functional determinant $|\partial f_r/\partial y_s|$. Analogous results hold in the domain of real variables.

30. Professor R. E. Langer: *The asymptotic location of the roots of a certain transcendental equation.*

The asymptotic location of the roots of the transcendental equation of the type $\sum_{i=1}^{n}\omega_i(\lambda)e^{\lambda B_i}=0$, in which the B_i are complex constants, has been studied by a number of authors under varying hypotheses on the coefficient functions $\omega_i(\lambda)$. In the most general case thus far treated—by Pólya and Schwengeler—these functions are taken to be polynomials. Wilder and Tamarkin had previously considered the case in which $\omega_i(\lambda)=\lambda^{r_i}\{a_i+E_i(\lambda)/\lambda\}$, the a_i being constants, the $E_i(\lambda)$ bounded, and the r_i being either all zero or else integers subject to special relations except when $n=2$. In this paper the equation is discussed for the general case $\omega_i(\lambda)=\lambda^{k_i}\{a_i+\epsilon_i(\lambda)\}$, where the k_i are any real constants and $\epsilon_i(\lambda)\to 0$, as $|\lambda|\to\infty$. The discussion is based on the papers of Tamarkin and Wilder.

31. Professor G. Y. Rainich: *A new proof of the theorem that the curvature field determines the electromagnetic field.*

Several years ago this theorem was proved by the author, using a vector method, for all cases with exception of the singular case where the electric and magnetic vectors are equal and at right angles (Transactions of this Society, vol. 27, p. 106). In the present paper the proof is given in coordinates and the theorem is shown to be true also in the case which was left out before.

32. Dr. H. H. Germond: *Solution of a non-linear equation of diffusion under specified boundary conditions.*

The problem of the moisture distribution in an initially soaked block of wood drying from one face involves the solution of a non-linear equation of diffusion. The substitution $z=b+x/(\alpha t)^{1/2}$ reduces the partial differential equation to an ordinary differential equation. A solution satisfying the boundary conditions is obtained as a power series in half-powers of z.

A simple approximate expression is obtained for the moisture distribution. The rate of moisture loss is deduced, and is compared with experimentally determined values.

33. Mr. N. H. McCoy: *Some commutation formulas in the algebra of quantum mechanics.*

For one pair of conjugate variables the special character of this algebra is due to the assumption $pq-qp=h/(2\pi i)=cI$, where q and p are matrices representing the coordinate and momentum respectively and I is the unit matrix. It has been shown by the author (Proceedings of the National Academy, March, 1929) that if f and ϕ are arbitrary polynomials in p and q,

$$f\phi - \phi f = \sum_{s=1}^{\infty}(c^s/s!)\left[\partial^s\phi/\partial q^s \cdot \partial^s f/\partial p^s - \partial^s f/\partial q^s \cdot \partial^s\phi/\partial p^s\right].$$

This formula is extended to polynomials in n pairs of variables. An application of the formula for one pair of variables to the problem of expanding functions such as $q^m p^{m'} q^n p^{n'}$, isobaric in p and q, in the form $\sum a_i p^i q^i$ gives rise to some interesting identities among binomial coefficients.

34. Mr. N. H. McCoy: *On vectors whose components are quantum variables.*

This paper deals with the rules of combination of vectors whose components are non-commutative quantum variables. Some analogs of the formula $pf-fp=c\partial f/\partial q$ are obtained for scalar and vector products. General commutation formulas for scalar and vector products of any two vectors are obtained by means of the results of the preceding paper.

35. Professor T. A. Pierce: *Symmetric functions of n-ic residues* (mod p).

This paper will appear in full in an early issue of this Bulletin.

36. Dr. T. W. Moore: *On the resultant of two binary forms.*

This paper contains a detailed discussion of the application of the method of elimination proposed by Morley and Coble (American Journal, vol. 49 (1927), pp. 463–488) to the case of two binary forms. The forms which the resultant determinant can take in the different cases are explained and a comparison drawn with the known methods of elimination. A non-symbolic method for writing down the resultant determinant by use of a multiple summation is furnished.

37. Professor E. G. Keller: *Periodic orbits about an anchor ring.*

This paper proves the existence of and constructs periodic orbits of the following types about a circular anchor ring, the radius of whose aperture and meridian sections are respectively $R-\epsilon$ and ϵ. Only orbits re-entrant

after one revolution are considered. Let (r, θ, z) and (ρ, θ, ϕ) be the cylindrical and polar coordinates of the particle. Orbits exist in the meridian plane of the form $r = \rho \cos \phi$, $z = \rho \sin \phi$, where

$$\rho = \sum_{i=0}^{\infty} \rho_i(t)\epsilon^i, \qquad \phi = \sum_{i=0}^{\infty} \phi_i(t)\epsilon^i$$

and where $\rho_i(t)$ and $\phi_i(t)$ are sums of sines and cosines of only even multiples of the independent variable t. These orbits are symmetrical with respect to both the equatorial plane and the axis of the ring. Orbits in the equatorial plane are expressible as power series in ϵ with trigonometric coefficients in t. Orbits inclined to the equatorial plane are of the form

$$r = \sum_{i=0}^{\infty} r_i(t)\epsilon^i, \qquad z = \sum_{i=1}^{\infty} z_i(t)\epsilon^i,$$

where $r_i(t)$ and $z_i(t)$ are either trigonometric or elliptic functions of t. The differential equations of motion of the particle admit one integral of areas and the vis viva integral, both of which are expressed in a finite number of terms.

38. Professor W. E. Milne: *On the degree of convergence of the Gram-Charlier Series.*

The author studies the series $\phi(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + \cdots$, where $\phi_0(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and where ϕ_n is the nth derivative of ϕ_0. It is shown that if $\phi(x)$ has a continuous kth derivative of bounded variation and if $x^m e^{x^2/4}\phi^{(k+1-m)}(x)$ $(m = 1, 2, \cdots, k+1)$ are of bounded variation in the interval $(-\infty, \infty)$, then the sum of the first n terms of the series will differ from $\phi(x)$ by a quantity which is $O(n^{-k/2})$. A similar theorem, but with lighter restrictions at infinity, is obtained for the series of orthogonal functions $f(x) = c_0w_0(x) + c_1w_1(x) + c_2w_2(x) + \cdots$ in which $w_n = e^{x^2/4}\phi_n$.

39. Professor C. G. Latimer: *On identities in theta-functions.*

In a former paper, the writer determined the sets of integral elements in the generalized quaternion algebras when both parameters are odd, the definition of *integral element* being due to Dickson. In this paper it is shown that Jacobi's fundamental identity in θ-functions may be proved by employing Hurwitz' integral quaternions, and the same method is applied to the above mentioned quaternion algebras to obtain new identities in θ-functions.

M. H. INGRAHAM,
Associate Secretary

THE USE OF CONTINUED FRACTIONS IN THE DESIGN OF ELECTRICAL NETWORKS*

BY T. C. FRY

1. *Introduction.* The literature on the subject of electrical circuit theory deals very largely with one type of problem: "Given an electrical system and the driving force which is imposed upon it, to find how it responds."

The converse problem, namely "Given a known driving force and the response which it is desired to produce, to find a system which will so respond," is much less frequently dealt with in the literature. It is this problem with which the present paper is to deal.†

About the electrical side of the problem we need only say that, whatever the language in which the engineer may phrase his requirements, they can always be satisfied provided we can produce one or more networks of preassigned *impedances*. It is, then, no serious restriction upon the prac-

* An address presented at the request of the program committee at the meeting of this Society at Cincinnati, December 1, 1928.

† Those who are interested in following the earlier literature will find the following references helpful. They will also give some idea of the technical problems to which the theory is applicable.

Fry, U. S. Patent #1,570,215, 1926, filed June 1921.

Zobel, *Theory and design of uniform and composite electric wave filters*, Bell System Technical Journal, vol. 2 (1923), p. 1.

Bartlett, *A note on the theory of artificial telephone and transmission lines*, Philosophical Magazine, vol. 48 (1924), p. 859.

Bartlett, *Properties of the generalized artificial line*, Philosophical Magazine, vol. 1 (1926), p. 553.

Cauer, *Die Verwirklichung von Wechselstromwiderständen vorgeschriebener Frequenzabhängigkeit*, Archiv für Elektrotechnik, vol. 17 (1926), pp. 355–388.

Bartlett, British Patent #290,701, 1928, filed January 1927.

Mead, *Phase distortion and phase distortion correction*, Bell System Technical Journal, vol. 7 (1928), p. 195.

Zobel, *Distortion correction in electrical networks with constant resistance recurrent networks*, Bell System Technical Journal, vol. 7 (1928), p. 438.

ticability of our theory from a technical standpoint to confine ourselves entirely to the design of networks with given impedance properties.

Our point of departure is a very simple observation regarding the ladder-type network shown in Fig. 1. The ordinary laws of circuit theory lead immediately to the formula*

$$F(p) = J_1 + \cfrac{1}{Y_2 + \cfrac{1}{J_3 + \cfrac{1}{Y_4 + \cdots}}} \tag{1}$$

If, then, the various J's and Y's are known, we can write down $F(p)$ at once. For example, if all the series elements

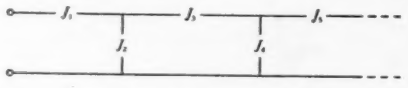

FIG. 1

are resistances $r_1, r_3, \cdots$, and all the shunt elements capacities $c_2, c_4, \cdots$, (1) will have the form of the Stieltjes fraction

$$F(p) = r_1 + \cfrac{1}{c_2 p + \cfrac{1}{r_3 + \cdots}} \tag{2}$$

Conversely, if we are given the problem of designing a network the impedance of which is to be some known function of frequency, and if we can expand this function $F(p)$ in a continued fraction of the Stieltjes type, we can certainly furnish the required network; for we need only cause the successive resistances in our ladder structure to be equal to the successive constant terms of the fraction, and the successive capacities to the coefficients of the corresponding terms in p.

* The admittances Y_i are reciprocals of the corresponding impedances J_i.

We can, then, apply the entire theory of Stieltjes fractions directly to the problem of network design. But there are many physical networks—even many networks of the ladder type—whose impedances cannot be expanded in the form (2), though they can be expressed in the form (1) if the J's and Y's are not restricted to be constants and multiples of p. For example, the simple function†

$$F^*(p) = \frac{2p + 3}{p + 2} \tag{3}$$

cannot be represented by a Stieltjes fraction, though it can be written in the form

$$F^*(p) = 1 + \cfrac{1}{1 + \cfrac{1}{p + 1}},$$

which is quite suitable for purposes of design. The Stieltjes fraction (2), in other words, is too special for our needs.

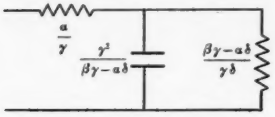

FIG. 2

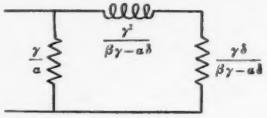

FIG. 3

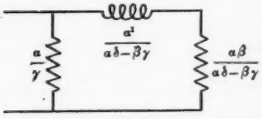

FIG. 4

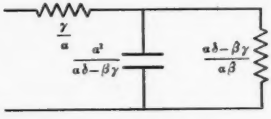

FIG. 5

In the present paper we shall obtain the necessary degree of generality by allowing the J's and Y's of (1) to be of the bilinear form

$$\frac{\alpha p + \beta}{\gamma p + \delta},$$

† The reason for the asterisk will appear later.

α, β, γ and δ being *positive* constants. Such a bilinear form can always be assigned a definite design significance. For if $\beta\gamma > \alpha\delta$, it can be interpreted as the impedance of the structure shown in Fig. 2 and the admittance of that shown in Fig. 3; while if $\beta\gamma < \alpha\delta$, it can be interpreted as the impedance of Fig. 4, or the admittance of Fig. 5.

2. *Resumé of Theorems about Stieltjes Fractions.* Obviously, since the Stieltjes fraction is so closely related to our problem, we must have the essential facts concerning it before us. They are as follows.

THEOREM 1. *A Stieltjes fraction*

$$f(z) = \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cdots}}} \tag{4}$$

converges for every value of the complex variable z, except perhaps certain negative real values, provided the a's are positive reals, and provided $\sum a_i$ *diverges.**

THEOREM 2. *The function f(z) defined by a convergent Stieltjes fraction can always be expressed in the form*

$$f(z) = \int_0^\infty \frac{d\Phi(x)}{z + x}, \tag{5}$$

where $\Phi(x)$ *is a monotonic non-decreasing real function of the real variable x, where* $\Phi(0) = 0$ *and* $\Phi(\infty) = 1/a_1$. *It is not necessary that* $\Phi(x)$ *be continuous.*†

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* Stieltjes, Œuvres Complètes, vol. 2, p. 465. If Σa_i converges, the convergents of even and odd orders separately approach limits $f_1(z)$ and $f_2(z)$, but the limits are not equal. As we have no interest in this case we shall give no further attention to it.

The case of complex *a*'s was afterward discussed by Van Vleck, Transactions of this Society, vol. 2 (1901), pp. 262, 344. Such coefficients have no meaning in the problem of network design, and will therefore not be considered here.

† Stieltjes, loc. cit., pp. 491, 493.

THEOREM 3. *The function $f(z)$ defined by a convergent Stieltjes fraction is therefore a regular analytic function, except at certain points of the negative real axis,* and is real for positive real values of z. The same is true of its reciprocal.*†

THEOREM 4. *The function $\Phi(x)$ in* (5) *is related to $f(z)$ by the law*

$$\Phi(x) = \frac{1}{2\pi i}\int f(z)dz, \tag{6}$$

the path of integration extending along a circle of radius x about the origin, beginning at $-x-i0$ and ending at $-x+i0$.‡

THEOREM 5. *Conversely, if $\Phi(x)$ is any monotonic non-decreasing real function of x in the "finite" interval $(0, b)$, and constant for $x \geqq b$, the function $f(z)$ defined by* (5) *possesses a convergent Stieltjes expansion with positive real coefficients.*§

THEOREM 6. *If $\Phi(x)$ is any monotonic non-decreasing function of x, such that the integrals*

$$\int_0^\infty x^n d\Phi(x) \tag{7}$$

all exist, the function $f(z)$ defined by (5) *possesses a "formal" Stieltjes expansion with positive real coefficients, which, if it converges at all, converges to the value $f(z)$ for all values of z except those on the negative real axis.*‖

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* Perron, *Die Lehre von den Kettenbrüchen*, p. 369.

† This is readily seen as follows: If (4) converges to $f(z)$, the fraction

$$\cfrac{1}{a_2 x + \cfrac{1}{a_3 + \cdots}}$$

must converge to $f_1(z) = 1/[zf(z)] - a_1$; for the convergents of this latter fraction are the corresponding convergents of the former one, divided by z and reduced by a_1. The new fraction, however, is a Stieltjes fraction, and therefore $f_1(z)$, and hence also $1/f(z)$, is regular everywhere except on the negative real z-axis.

‡ This is a slight modification of the theorem as given by Perron, loc. cit., p. 372.

§ Perron, loc. cit., p. 388.

‖ Stieltjes, loc. cit., p. 504.

Thus, if we are given a function $f^*(z)$ we can determine whether or not it possesses a Stieltjes expansion by the following systematic procedure:

(a) First we observe whether it is real for positive real values of z. If not, no usable expansion is possible.

(b) Next we investigate the singularities of $f^*(z)$ and its reciprocal. Unless they are all on the negative real axis no expansion is possible.

(c) If they are all on the negative real axis we compute a function $\Phi(x)$ from the formula

$$\Phi(x) = \frac{1}{2\pi i}\int f^*(z)dz, \tag{8}$$

the path of integration being that defined in Theorem 4. Unless it is monotonic non-decreasing, no expansion is possible. Even if it is monotonic non-decreasing no expansion is possible unless either $\Phi(x)$ is a finite constant for $x>b$, or else the integrals (7) all exist.

(d) If the possibility of expansion is still not disproved, we substitute our $\Phi(x)$ in (5) and evaluate the integral. The $f(z)$ which results may or may not be the function $f^*(z)$ with which we started. If it is not, $f^*(z)$ can neither be expressed in the form (5), nor expanded in a Stieltjes fraction. But if $f(z)$ is identical with $f^*(z)$, we know either that a convergent expansion *does* exist, or that it *may* exist, according as we are working under the conditions of Theorems 5 or 6.†

We may, then, fail to get a Stieltjes expansion for either of four reasons:

(a) Because $f^*(z)$ is not real when z is real and positive;

† We say nothing about the processes by means of which the coefficients are obtained after we have shown that an expansion is possible, for the reason that these processes play no part in our study. They are tedious, but otherwise quite satisfactory. They will be found in almost any discussion of continued fractions, the most satisfactory references being Stieltjes, Œuvres Complètes, vol. 2, p. 184, for the more familiar method, and Thiele, *Interpolationsrechnung*, p. 138, for an interesting alternative.

(b) Because the singularities of $f^*(z)$ or its reciprocal are not properly located;

(c) Because $\Phi(x)$ is not of an allowable type;

(d) Because, though $f^*(z)$ gives rise to a $\Phi(x)$ of suitable type, it is still not true that

$$f(z) = \int \frac{d\Phi(x)}{z + x}$$

is equal to $f^*(z)$.

We shall consider these difficulties in order, seeking in each case to find a means of deriving a type of fraction which can be interpreted electrically.

We shall occasionally find it desirable to have separate names for the functions $f^*(z)$ and $f(z)$. The former of these—that is, the function which we wish to represent in fractional form, and from which we obtain $\Phi(x)$ by the use of (6)—we shall call the *generating function*. The other—that is, the function to which the fraction converges if we succeed in getting a convergent fraction at all, and which is obtained from $\Phi(x)$ by the use of (5)—will be said to *correspond* to $\Phi(x)$. The important distinction is, that while many functions may generate the same $\Phi(x)$, only the one which corresponds to it is capable of Stieltjes representation.

3. *Redistribution of Singularities by Change of Variable.* The first difficulty is inherent in the physical problem. For if a physical system had an impedance $Z(p)$ which was complex for a real positive $p = p_0$, a *real* driving force which varied with time according to the law $e^{p_0 t}$ would produce a *complex* current. This is absurd, since a complex current has no physical meaning.

Turning next to the second difficulty, we first note that physically possible systems cannot have singularities in the right-hand half of the plane. This is easily seen in the case of poles, for a pole would imply that once an oscillation of the proper frequency were established it would persist forever *with constantly increasing amplitude* even in the absence of any sustaining force, which is obviously absurd.

It is not so easy to exclude the possibility of essential singularities, but they are never met.

On the other hand, singularities in the left-hand half of the plane—or as a limiting case on the imaginary axis—are physically possible. In some simple cases they can be shifted to the negative real axis by a suitable change in variable, and an expansion thus obtained.

Thus, suppose we are given a function $F^*(p)$ which, together with its reciprocal, has only pure imaginary singular points. Then by the transformation

$$z = p^2, \tag{9}$$

we will arrive at a function $f^*(z) \equiv F^*(z^{1/2})$, whose singularities occur at negative real values of z. From this function we may be able to obtain a Stieltjes fraction of the form (4), and if so the equivalent fraction in p,

$$F(p) \equiv f(p^2) = \cfrac{1}{a_1 p^2 + \cfrac{1}{a_2 + \cdots}} \quad ,$$

will converge for all values of p except possibly certain pure imaginaries.

Even though this $F(p)$ proved to be identical with $F^*(p)$, however, it would not be suitable for the purposes of circuit design; for we have no type of circuit element whose impedance or admittance varies as the square of the frequency. However, if we were to set $f^*(z) \equiv F^*(z^{1/2})/z^{1/2}$, we might arrive at a usable expansion.† For suppose we succeeded in finding a convergent fraction (4) the limit of which was identical with this $f^*(z)$. We would then have at once

$$F^*(p) = \cfrac{1}{a_1 p + \cfrac{1}{a_2 p + \cfrac{1}{a_3 p + \cdots}}} \quad .$$

† We have probably introduced a new singularity at $z = 0$ by the introduction of the new factor, but as this is an end point of the negative real axis, it is not objectionable.

This form of development is altogether satisfactory. It corresponds to a ladder network all the series elements of which are inductances and all the shunt elements condensers.

Similarly, if the singularities of $F^*(p)$ and its reciprocal all lie on a line parallel to the imaginary axis and α units to the left of it, we may transfer them all to the negative real axis by either of the transformations†

$$z = (p + \alpha)^2, \tag{10}$$

or

$$z = p^2 + 2\alpha p. \tag{11}$$

It is easily seen that either leads to a usable type of fraction provided we identify $f^*(z)$, not with $F^*(p)$, but rather with $F^*(p)/(p+\beta)$, where $0 \leqq \beta \leqq 2\alpha$.‡

4. *Fractions which Represent Functions in Island Regions.* The function $f^*(z)$ obtained from $F^*(p)$ by such transformations as (9), (10) and (11) is ordinarily multiple-valued.

On the other hand, the function defined by any fraction of the form (4) must be single-valued. Hence if we succeed in obtaining such a fraction, it will represent one branch of $f^*(z)$ only. Consequently, when rewritten in terms of p, it will represent $F^*(p)$ over only that portion of the p-plane which corresponds to this branch of $f^*(z)$. In the remainder of the p-plane it will converge (except, perhaps, at certain points of the boundary which separates the two regions), but usually not to the value $F^*(p)$.

The fraction, in other words, will represent $F^*(p)$ only in an "island region" (espace lacunaire). For example, by using the transformation $z = p^2 + p$ we can expand the function (3) into the form

† The second of these transformations has a wider field of use than the first, for it carries to the negative real axis of z not only the p's on the vertical line mentioned, but p's in the *real* range $(-2\alpha, 0)$ as well. Singularities within this real interval are therefore also allowable.

‡ Since $p = -\beta$ corresponds to a negative real z, the new singularity at this point is not objectionable.

$$(12)\qquad F(p) = 1 + \cfrac{1}{\cfrac{2}{p+1} + \cfrac{1}{\cfrac{1}{p} + \cfrac{1}{\cfrac{1}{p+1} + \cdots}}},$$

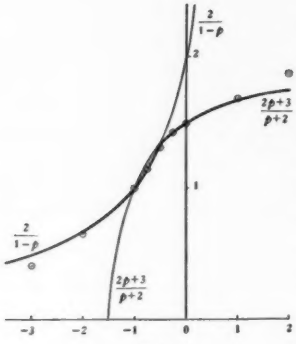

FIG. 6

the remaining terms being alternately $1/p$ and $1/(p+1)$. This fraction converges for every value of p, real or complex, except certain values with the real part $-1/2$. To the right of this dividing line the limit is actually equal to $F^*(p) = (2p+3)/(p+2)$. To the left, it is $2/(1-p)$. Fig. 6 shows the graphs of these functions for real values of p, together with certain points on the fifth convergent of the fraction (12). It is obvious that, in the region covered by the graph, convergence is quite rapid.

Now the outstanding thing about this expansion is, that its singular line bears no relation to $F^*(z)$ whatever, but is determined by the character of the transformation instead. Had we begun with any other function and used the same transformation, and had we succeeded in obtaining a convergent fraction for this new function, the new fraction also would have had this same line for a cut, and would have been regular everywhere else.

5. *Properties of a Particular Transformation.* These observations at once suggest two methods of dealing with functions the singularities of which are not on the negative real axis. One is, to search for a transformation which will carry the singularities to the negative real axis. The other is, to search for a transformation which will carry them all

into *one* branch of a multiple-valued function of z, thus leaving in the *other* branch only the singularities of the transformation itself. In either case, of course, we must require that the latter singularities shall lie on the negative real z-axis. The second of these methods is obviously the more powerful one, and we shall find that an especially suitable transformation is

$$z = \frac{(ap + b)(\alpha p + \beta)}{(cp + d)(\gamma p + \delta)}, \tag{13}$$

where the coefficients $a, b, c, d, \alpha, \beta, \gamma, \delta$ are all positive reals.

It requires only routine algebra to establish the following facts:

(1) In order that the singular points of the transformation (13) shall be on the negative real z-axis it is necessary and sufficient that both

$$\begin{cases} (b\gamma - a\delta)(c\beta - d\alpha) > 0, \\ (bc - ad)(\beta\gamma - \alpha\delta) > 0. \end{cases} \tag{14}$$

(2) The p-transformed of the negative real z-axis consists of part of the negative real p-axis together with a circle whose center is at

$$p_0 = -\frac{\beta\gamma(bc - ad) + ad(\beta\gamma - \alpha\delta)}{\alpha\gamma(bc - ad) + ac(\beta\gamma - \alpha\delta)},$$

and whose radius is

$$r^2 = p_0^2 - \frac{\beta\delta(bc - ad) + bd(\beta\gamma - \alpha\delta)}{\alpha\gamma(bc - ad) + ac(\beta\gamma - \alpha\delta)}.$$

We shall call this circle the "boundary of the transformation."

(3) So long as (14) is satisfied, p_0 is negative and $r^2 \leqq p_0^2$. Hence the boundary of the transformation never crosses the imaginary axis, though in a limiting case it may be tangent to it at the origin. It may also degenerate into a line parallel to, and to the left of, the imaginary axis; or into the imaginary axis itself. We may therefore state the following theorem.

THEOREM 7. *Any transformation of the type* (13) *which has its singularities located on the negative real z-axis, and for which all the constants* $a, b, c, d, \alpha, \beta, \gamma, \delta$ *are positive or zero, maps a circular area in the left half of the p-plane upon one sheet of the z-plane, the remainder of the p-plane being mapped on the other sheet of the z-plane. The circular boundary may be tangent to the imaginary p-axis, or it may degenerate into a vertical straight line, but it cannot in any case enclose a portion of the right-hand half of the plane.*

In view of these conclusions we can rewrite (13) in one or the other of three convenient forms.* The first is

$$z = \frac{r^2 p^2}{\lambda^2 (p+s)^2 - p^2}, \qquad s \geqq 0, \quad \lambda \geqq 1, \tag{15}$$

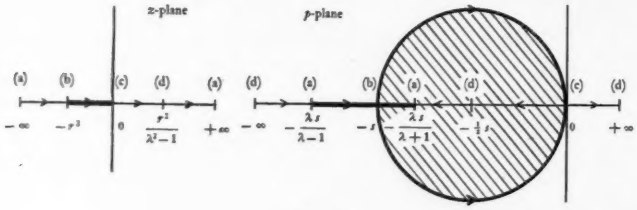

FIG. 7

which represents the configuration shown in Fig. 7. The boundary of the transformation in the p-plane, and the portion of the negative z-axis between the branch points, are drawn heavy. The letters (a), (b), $\cdots$, in the two drawings represent corresponding points, and the numbers written below them are their coordinates. Thus (a) is the point $z=\infty$, and transforms into $p=-\lambda s/(\lambda \pm 1)$. The singular points of the transformation are at (b) and (c).

* It will be noted that (16) is equivalent to an inversion in p, followed by (15); also that (17) is equivalent to a translation in p, followed by (15), and then by a translation in z. The set of transformations is closed under inversions in z.

By setting $\lambda=1$, $\mu=\alpha$, $r=s=\infty$, and $\nu=0$ or α in (17), we get (10) and (11).

The second form is

$$z = \frac{r^2}{\lambda^2(sp+1)^2 - 1}, \qquad \begin{matrix} s \geqq 0, \\ \lambda \geqq 1, \end{matrix} \tag{16}$$

which represents the configuration shown in Fig. 8. In this case the boundary of the transformation consists of a part of the negative p-axis, together with the vertical line through $p = -1/s$. The singular points of the transformation are at (b) and (c).

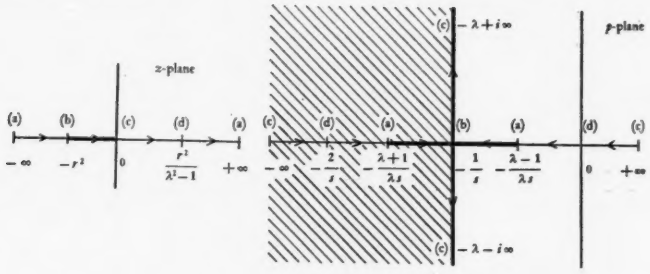

FIG. 8

The third form is

$$z + \nu = \frac{r^2(p+\mu)^2}{\lambda^2(p+\mu+s)^2 - (p+\mu)^2}, \qquad \begin{matrix} s \geqq 0, \\ \lambda \geqq 1, \end{matrix} \tag{17}$$

$$0 \leqq \nu \leqq \frac{r^2\mu^2}{\lambda^2(s+\mu)^2 - \mu^2},$$

for which the configurations are shown in Fig. 9. They are similar, except for the positions of the origins, to Fig. 7. The singular points are again at (b) and (c).

Finally, we notice that, if we had transformed $F^*(p)$ into a function of z by the use of (13), and if we had succeeded in obtaining a Stieltjes expansion for the latter, it would have been of the form (4), which becomes in terms of p

$$f(z) = \cfrac{1}{\cfrac{(ap+b)(\alpha p+\beta)}{a_1(cp+d)(\gamma p+\delta)} + \cfrac{1}{a_2 + \cdots}}.$$

This form in itself is not suitable for design purposes. But

$$(18)\quad \frac{(ap+b)}{(cp+d)}f(z) = \cfrac{1}{a_1\dfrac{\alpha p+\beta}{\gamma p+\delta} + \cfrac{1}{a_2\dfrac{ap+b}{cp+d} + \cfrac{1}{a_3\dfrac{\alpha p+\beta}{\gamma p+\delta} + \cdots}}}$$

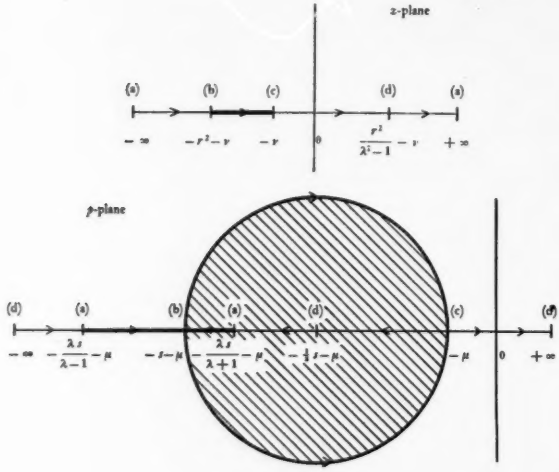

FIG. 9

is suitable for such uses. So also is

$$(19)\quad \frac{\alpha p+\beta}{\gamma p+\delta}f(z) = \cfrac{1}{a_1\dfrac{ap+b}{cp+d} + \cfrac{1}{a_2\dfrac{\alpha p+\beta}{\gamma p+\delta} + \cfrac{1}{a_3\dfrac{ap+b}{cp+d} + \cdots}}}$$

Hence, in using the transformation (13) for design purposes we would usually† identify $f^*(z)$ with the transformed of

† These are not the only *possible* bilinear multipliers which can be successfully used. However, we shall see that they serve our purpose remarkably well.

either $(cp+d)F^*(p)/(ap+b)$ or $(\gamma p+\delta)F^*(p)/(\alpha p+\beta)$ in order that $F^*(p)$ might itself appear in one of the forms (18) or (19). The additional factors introduce no objectionable singularities into $f^*(z)$, since the vanishing of either $ap+b$, $cp+d$, $\alpha p+\beta$ or $\gamma p+\delta$ leads to either $z=0$ or $z=\infty$, both of which are "negative real values of z." Inspection shows, moreover, that unless these points are essential singularities, the one at $z=0$ must be a pole, and the one at $z=\infty$ a zero.

6. *Theorems Relating to Redistribution of Singularities.* Suppose, now, that we are given a function $F^*(p)$ whose singularities, together with those of its reciprocal, are all more than $1/s$ units (s finite) to the left of the imaginary p-axis. Then by the use of the transformation (16), Fig. 8, we can obtain a function $f^*(z)$ with branch points at 0 and $-r^2$, *and no other singularities or zeros* in its one branch except possibly a simple zero at ∞. Moreover, the length r^2 of the cut in the z-plane may be finite.

If on the other hand we deal with Fig. 7 and suppose that either the given function $F^*(p)$ or its reciprocal is singular at $p=0$, but that all other singularities can be enclosed in the interior of a finite circle, it is a simple matter to show that all the essential facts of the last paragraph are again true. Hence we have the following theorem.

THEOREM 8. *From any $F^*(p)$ all the singularities of which, together with those of its reciprocal, lie at least $1/s$ units to the left of the imaginary p-axis, (or all the singularities of which, together with those of its reciprocal, except one at the origin, can be enclosed in the interior of a circle of finite radius), we can obtain a function $f^*(z)$ in at least two ways† which, together with its reciprocal, will be regular everywhere in the cut z-plane except at the two finite branch points 0 and $-r^2$, and except also for a simple zero at ∞. It has, moreover, at all points of the cut except the ends, a regular analytic continuation either from above or below.*

† Corresponding to the two methods of identification discussed in §3.

We shall, in this paper, have little to say about functions $F^*(p)$ which do not satisfy the conditions of Theorem 8, since they lead us into the ambiguities of Theorem 6. However, we state the following theorem without proof:

THEOREM 9. *If $F^*(p)$ and its reciprocal have no singular points to the right of the imaginary axis, but if the conditions of Theorem* 8 *can nevertheless not be met, the transformation* (13) *can still produce a function $f^*(z)$ the singularities of which, together with those of its reciprocal, are all located on the negative real axis. In this case, however, the entire negative z-axis is a cut, and the possibility of analytic continuation can in general not be asserted for every point on it.*

If $F^*(p)$ or its reciprocal has a singular point to the right of the imaginary p-axis no transformation of the form (13) *can carry it into an $f^*(z)$ the zeros and singularities of which all lie on the negative real axis.*

We have, then, found a way of overcoming the second of our difficulties; for we have found a method of creating, from any function $F^*(p)$ whose singularities are all to the left of, or on, the imaginary axis, another function $f^*(z)$ whose singularities are located as required by the Stieltjes theory; and this has been accomplished in such a way, moreover, that the existence of an expansion of the form (4) for $f^*(z)$ implies the existence of a usable expansion of the form (18) or (19) for $F^*(p)$ also. We now turn our attention to the third difficulty: the possibility that $f^*(z)$ may not generate a $\Phi(x)$ of suitable character.

7. *Existence Theorems concerning the Auxiliary Function* $\Phi(x)$. To begin with, we introduce a definition which will aid us in avoiding much circumlocution.

The function $f(w)$ is said to be "admissibly" singular at the point $w=w_0$, if it possesses there a simple pole and/or† *an essential singularity of such a character that*‡

† We do not wish to exclude such functions as $(w-w_0)^{1/2}+1/(w-w_0)$.

‡ It is obvious that this condition excludes isolated essential singularities. It includes many algebraic and logarithmic branch points, however.

$$(20) \qquad |(w - w_0)^k f(w)| < M, \quad 0 \leqq |w - w_0| < \delta,$$

where δ and M are any positive numbers other than 0 and ∞, and where $0 \leqq k < 1$.

We can then readily prove the following theorem.

THEOREM 10. *If the function $F^*(p)$ and its reciprocal have none but admissible singularities on the imaginary p-axis, the $f^*(z)$ obtained either from $[(cp+d)/(ap+b)]F^*(p)$ or from $[\gamma p+\delta)/(\alpha p+\beta)]F^*(p)$ by the use of* (13) *will have only admissible singularities on the negative real axis. The same is true of $1/f^*(z)$.*

At all but the branch points of the transformation this is immediately obvious, for the character of a singularity is never altered by a conformal transformation. At the singular points the transformation substantially replaces $p-p_0$ by $(z-z_0)^{1/2}$; whence even if the additional bilinear terms in p happen to be singular at such a point they cannot render $f^*(z)$ inadmissibly singular. We need hardly fill in the details of the proof.

Next we note that, since $F^*(p)$ is real for every real positive p, $f^*(z)$ is also real for every real positive z. This follows from a very simple study of analytic continuation in the p-planes of Figs. 7, 8 and 9. Then from an equally easy study of analytic continuation in the z-planes we find that $f^*(x+iy)$ and $f^*(x-iy)$ are conjugate complex. Hence, if we introduce the integral

$$(21) \qquad G(z) = \int_\alpha^z f^*(z)dz,$$

the integration being performed along any path which passes from the (arbitrary) positive real α to z without touching the negative real axis, we see at once that $G(x+iy)$ and $G(x-iy)$ are conjugate complex. But by (8)

$$(22) \qquad \Phi(x) = \lim_{y \to 0} \frac{G(-x+iy) - G(-x-iy)}{2\pi i}.$$

Hence if this limit exists at all it is real.

As for the existence of the limit, if $f^*(z)$ is regular and capable of analytic continuation across the axis, or if it possesses a singularity of the sort (20), $G(-x+iy)$ and $G(-x-iy)$ approach well-defined finite limits. If on the other hand $f^*(z)$ has a simple pole at $z=-x$, it may be written as $\alpha/(z+x)+f_1^*(z)$, where $f_1^*(z)$ has the same analytical character as $f^*(z)$ at all other points than $-x$, and at $-x$ satisfies (20). Then by actual integration

$$\Phi(x) = \frac{1}{2}\alpha + \lim_{y\to 0} \frac{G_1(-x+iy) - G_1(-x-iy)}{2\pi i},$$

where $G_1(z)$, being derived from $f_1(z)$, is known to have well-defined limits. Hence we have the following theorem.

THEOREM 11. *Any function $f^*(z)$ which is real for real positive z, regular for all but real negative z, and only admissibly singular for real negative z, generates a well-defined real function $\Phi(x)$.*

We are now prepared to state, as a consequence of Theorems 8, 9, 10 and 11, the following theorem, which has been the objective of the present section.

THEOREM 12. *From any function $F^*(p)$ which is real for positive real values of p, which has no singularities to the right of the imaginary axis, and none but admissible singularities on the imaginary axis, at least two functions $f^*(z)$ can be produced by the use of the transformation* (13), *each of which will generate a $\Phi(x)$ which is well-defined for every x.*

If, in addition, the conditions of Theorem 8 *are met by $F^*(p)$, $\Phi(x)$ will be of limited variation, and constant for $x>r^2$.*

Only the last paragraph of the theorem requires proof. That $\Phi(x)$ is constant for $x>r^2$ follows at once from (8) and from the fact, stated in Theorem 8, that $f^*(z)$ is regular for $|z|>r^2$.

To prove that $\Phi(x)$ is of limited variation we exclude for a moment the possibility of poles at the branch points, con-

sidering only singularities of the type (20). Then, since $f^*(z)$, and consequently $G(z)$ also, is regular as we approach the negative real axis from above and below, we have

$$\frac{d\Phi}{dx} = \lim_{y \to 0} \frac{1}{2\pi i} \frac{d}{dx}[G(-x+iy) - G(-x-iy)]$$

$$= \frac{1}{2\pi i} \lim_{y \to 0} [f^*(-x-iy) - f^*(-x+iy)] \tag{23}$$

for all values of x except $x=0$ and $x=r^2$. Hence the variation of $\Phi(x)$ is

$$\text{Var } \Phi = \int_0^{r^2} \left|\frac{d\Phi}{dx}\right| dx \leqq \frac{1}{\pi} \int_0^{r^2} |f^*(-x)|\, dx.$$

We now consider the half of this interval nearest the origin. We know that, over the range $0 < x \leqq \delta$, (20) is satisfied. We also know that $|f^*(z)|$ is bounded in the interval $\delta \leqq x \leqq r^2/2$. Call its upper bound m. Then if we take M_1 as the larger of the two quantities M and $(2/r^2)^k m$, it will be true that

$$|f^*(-x)| \leqq M_1/x^k$$

over the entire interval $0 < x \leqq r^2/2$. A similar statement (with $M = M_2$) can be made for the other half of the interval, $r^2/2 \leqq x < r^2$. Hence

$$\text{Var } \Phi \leqq (M_1 + M_2)(r^2/2)^{1-k}/(1-k),$$

which proves our theorem, provided the singularities of $f^*(z)$ are of the type (20).

If, on the other hand, $f^*(z)$ possesses a pole at either $z=0$ or $z=-r^2$, or both, these poles will have no other effect then to introduce finite discontinuities into $\Phi(x)$ equal to their residues; and it is seen at once that the theorem will still be true.

Let us now take stock of our study from the standpoint of network design. We know that, from any $F^*(p)$ which satisfies the conditions of Theorem 8, we can obtain an $f^*(z)$ the singularities of which are on the negative real z-axis, and

which generates a $\Phi(x)$ the variation of which is bounded, and which is constant for $x>r^2$. We may, therefore, replace $\Phi(x)$ by the difference of two functions, $\Phi_1(x)$ and $\Phi_2(x)$, both of which are monotonic, non-decreasing, and both finite for $x>r^2$. Hence, by Theorem 5, the functions $f_1(z)$ and $f_2(z)$ which correspond to $\Phi_1(x)$ and $\Phi_2(x)$ possess Stieltjes fractions. That is, when the conditions of Theorem 8 are satisfied, $f^*(z)$ generates a $\Phi(x)=\Phi_1(x)-\Phi_2(x)$, to which corresponds an $f(z)=f_1(z)-f_2(z)$, which can be written as the *difference* of two Stieltjes fractions.

As it stands, this result has no practical utility, for we cannot connect two networks in such a way that their combined impedance is the difference of their separate impedances. If, however, we can find a way of obtaining a fraction with positive coefficients the $\Phi(x)$ corresponding to which is monotonic *non-increasing*, [say a fraction the limit of which is $f_2'(z)$], then $f'(z)=f_1(z)+f_2'(z)$ will also generate the same $\Phi(x)$ as $f^*(z)$, and may turn out to be identical with it. Such a result would be usable. This leads us, then, to the search for a form of fraction the $\Phi(x)$ of which is monotonic non-increasing.

8. *Functions for which $\Phi(x)$ is Monotonic Decreasing.* We can easily show that if $f^*(z)$ gives rise to a monotonic *non-increasing* $\Phi(x)$, certain related functions give rise to monotonic *non-decreasing* Φ's. Chief among these are $zf^*(z)$, $1/f^*(z)$ and $f^*(1/z)$.

We treat first the case of $zf^*(z)$. Wherever $f^*(z)$ is regular as it approaches the negative real axis from above and below, $f_1(z)=zf^*(z)$ is also, and therefore either $f^*(z)$ or $f_1^*(z)$ can be substituted in (23). Thus we obtain

$$\frac{d\Phi_1(x)}{dx}=\frac{1}{2\pi i}\lim_{y\to 0}\left[f_1^*(-x-iy)-f_1^*(-x+iy)\right]$$

$$=\frac{-x}{2\pi i}\lim_{y\to 0}\left[f^*(-x-iy)-f^*(-x+iy)\right]$$

$$-\frac{1}{2\pi}\lim_{y\to 0} y\left[f^*(-x-iy)+f^*(-x+iy)\right].$$

The last limit, however, vanishes owing to the finiteness of $*(-x \pm iy)$. Hence by (23) we have

$$\frac{d\Phi_1(x)}{dx} = -x\frac{d\Phi}{dx}$$

or

$$d\Phi_1(x) = -x\,d\Phi(x). \tag{24}$$

On the other hand, at a pole the residue of $f_1^*(z)$ is $-x$ times the residue of $f^*(z)$. That is, (24) is still true.

The extension to singularities of the type (20) is almost equally simple. It consists merely in showing that at such a point both $\Phi(x+\delta x) - \Phi(x-\delta x)$ and $\Phi_1(x+\delta x) - \Phi_1(x-\delta x)$ vanish with δx. There is, then, no discontinuity at the point, and (24) rules at every neighboring point. Hence we may state the following theorem.

THEOREM 13. *If the singularities of $f^*(z)$ are all on the negative real axis and are of admissible type, and if it generates a monotonic function $\Phi(x)$, then the function $f_1^*(z) = zf^*(z)$ generates a function $\Phi_1(x)$ which is monotonic in the opposite sense. More specifically, the two are related by equation* (24).

We could frame somewhat similar proofs to show that the Φ's generated by $f^*(1/z)$ and $1/f^*(z)$ are monotonic in the opposite sense to that generated by $f^*(z)$ itself. However we have no occasion to use them in the present paper.

The significance of Theorem 13 for our purposes lies in the fact that, if $f^*(z)$ possesses a Stieltjes expansion of the form (4), $f_1^*(z) = zf^*(z)$ will have the expansion

$$f_1^*(z) = \cfrac{1}{a_1 + \cfrac{1}{a_2 z + \cfrac{1}{a_3 + \cdots}}} \tag{25}$$

This form of expansion, therefore, is appropriate for functions which generate monotonic decreasing Φ's. Moreover, in view of the remarks made at the end of §7, if we were given a

function of bounded variation, we would naturally seek to expand it in the sum of two fractions, one of form (4) and one of form (25).

9. *The Difference Function,* $\delta(z)$. We now have methods for overcoming, to a large degree, our second and third difficulties. But while we are thus enabled to effect an expansion of

$$f(z) = \int_0^b \frac{d\Phi(x)}{z + x}$$

in a usable form, we still do not know that this $f(z)$ and $f^*(z)$ are identical. We must now study this fourth difficulty.

If we write

$$\delta(z) = f^*(z) - f(z),$$

we know that

$$\frac{1}{2\pi i} \int \delta(z)dz = 0, \tag{26}$$

the integral being taken about any (every) circle of radius x about the origin. We ask what sort of function $\delta(z)$ can be. We see at once that

(a) *It is regular except, perhaps, at negative real values of* z. This follows from the fact that it is the difference of two regular functions.

(b) *It may be an entire function if* $f^*(z)$ *is singular at infinity; or if* $f^*(z)$ *is regular at infinity* $\delta(z)$ *may be a constant.* That any entire function satisfies the requirement (26) is obvious. However, if $f^*(z)$ is regular at infinity, $\Phi(x)$ is constant for x greater than some finite value r^2; $f(z)$ is therefore also regular for $|z| > r^2$.† Hence the same will be true of $\delta(x)$ as well.

(c) *It may have any number of isolated singular points the contour integrals about which are zero. Such singularities, however, can only occur where* $f^*(z)$ *is itself singular.* For example,

† Perron. loc. cit., p. 369.

$$\delta(x) = \frac{1}{(z+x)^2},$$

which has residue zero, is possible. To establish that $\delta(z)$ is regular wherever $f^*(z)$ is regular, we need only show that this is true of $f(z)$. Let us suppose, then, that $f^*(z)$ is represented on a Riemann surface of any number of sheets, connected along certain segments of the negative real axis, and that it has no singularities in the interval $(-x_2, -x_1)$. Elsewhere on the negative real axis it may be of any analytic character whatever consistent with the fact that it generates $\Phi(x)$. Then the $G(z)$ defined by (21) is regular for every z not on the negative real axis, and also for values of z between $(-x_2, -x_1)$; though in the case of this interval it may take different values $F_1(z)$ and $F_2(z)$ as we approach the axis from above and below. It follows, then, that

$$\Phi(z) = \frac{F_1(-z) - F_2(-z)}{2\pi i}$$

is a regular analytic function of z in the vicinity of (x_1, x_2). Along the real axis this function is, by definition, identical with $\Phi(x)$. Then by a well known theorem†, it is also true that

$$\int_{x_1}^{x_2} \frac{d\Phi(x)}{z+x} = \int_{x_1}^{x_2} \frac{d\Phi}{dx}\,\frac{dx}{z+x}$$

is regular—though perhaps multiple valued—between $-x_2$ and $-x_1$. On the other hand

$$\int_0^{x_1} \frac{d\Phi(x)}{z+x}$$

and

$$\int_{x_2}^{\infty} \frac{d\Phi(x)}{z+x}$$

are both analytic in this same region.‡ Since $f(z)$ is the sum of these three integrals, it too must be regular in the interval (x_1, x_2).

† Goursat, *Cours d'Analyse*, vol. 2, p. 254.

‡ Perron, loc. cit., p. 369.

(d) $\delta(z)$ *does not admit of non-essential cuts.* For suppose $(-x_2, -x_1)$ were such a cut: then it would be possible to effect analytic continuation not only of $\delta(z)$, but of the integral

$$\Delta(z) = \int_a^z \delta(z)dz$$

also, across this boundary. Let us call the value taken by $\Delta(z)$ as reached through the upper half of the plane $\Delta_1(z)$, and that reached through the lower half $\Delta_2(z)$. Then $\Delta_1 - \Delta_2$ possesses a well-defined set of derivatives, which we can obtain by choosing dz in any manner we like. For example, we may choose $dz = dx$ if we wish. But by (26) $\Delta_1 - \Delta_2$ is identically zero along the negative real axis; hence all the derivatives vanish. That is $\Delta_1(z) = \Delta_2(z)$, and the function $\delta(z)$ is its own analytic continuation across $(-x_2, -x_1)$.

As a final result of all these considerations we may enunciate the following theorem.

THEOREM 14. *Two functions $f_1(z)$ and $f(z)$ which are regular everywhere except on the negative real axis and generate the same $\Phi(x)$ can differ at most by a function $\delta(z)$ which is single-valued and regular, except perhaps at certain points where one or both of the functions f_1 and f are singular.*† *At such points $\delta(z)$ may possess isolated singularities of residue zero; or, in case the singularities of f_1 or f are everywhere dense in an interval $(-x_2, -x_1)$, the same may be true of δ also; but in this latter case the relation* (26) *will still be satisfied.*

A special case of peculiar interest is the type of function derived through the use of the transformation (13), when $F^*(p)$ satisfies the conditions of Theorem 8. In such cases, $f^*(z)$ has singularities at only $-r^2$ and 0; hence $\delta(z)$ can have singularities nowhere else. Usually it is found to be a constant.

† Our proof has assumed *one* function to *correspond* to $\Phi(x)$. But the difference between any two functions which generate the same $\Phi(x)$ is the sum of two δ's of the type with which our proof deals, whence the theorem is true as stated.

10. *An Example.* As an example of this process of expansion, let us consider

$$f^*(z) = (z + \tfrac{1}{2})(z^2 + z)^{1/2} - z^2 - z, \tag{27}$$

a function which is everywhere regular (including ∞) except at the winding points $z=0$ and $z=-1$. We wish to represent that branch of it for which the radical is positive for positive real z. We readily find that it generates

$$\begin{cases} \Phi(x) = -\dfrac{1}{3\pi}(x - x^2)^{3/2}, & 0 < x < 1, \\ \Phi(x) = 0, & 1 < x; \end{cases} \tag{28}$$

or, since it is the derivative† of Φ rather than Φ itself which is of interest,

$$\frac{d\Phi}{dx} = \frac{x}{\pi}(x - x^2)^{1/2} - \frac{1}{2\pi}(x - x^2)^{1/2}.$$

We can, then, divide Φ into a monotonic non-decreasing term Φ' and a monotonic non-increasing term Φ_1'' by choosing

$$\frac{d\Phi'}{dx} = \frac{x}{\pi}(x - x^2)^{1/2}$$

and

$$\frac{d\Phi_1''}{dx} = -\frac{1}{2\pi}(x - x^2)^{1/2}.$$

The function $f'(z)$ which corresponds to Φ' is found directly from (5) to be

$$\begin{aligned} f'(z) &= \int_0^1 \frac{d\Phi'}{dx}\,\frac{dx}{z + x} \\ &= z(z^2 + z)^{1/2} - z^2 - \frac{z}{2} + \frac{1}{8}. \end{aligned} \tag{29}$$

We know, then, that this function possesses a Stieltjes expansion.

† Since $\Phi(x)$ is differentiable, the Stieltjes integral $\int d\Phi/(z+x)$ reduces simply to $\int (d\Phi/dx)\,dx/(z+x)$.

To find the residual function $f_1''(z)$ which corresponds to Φ_1'', we adopt the process suggested by Theorem 13 and compute the auxiliary function

$$f''(z) = \frac{f_1''(z)}{z}$$

which corresponds to

$$\frac{d\Phi''}{dx} = -\frac{1}{x}\frac{d\Phi_1''}{dx} = \frac{1}{2\pi}\frac{(x^2 - x)^{1/2}}{x}.$$

Again, we find from (5) that

$$f''(z) = \frac{1}{2}\left(\frac{(z^2 + z)^{1/2}}{z} - 1\right).$$

This also possesses a Stieltjes expansion. Then

$$f_1''(z) = \tfrac{1}{2}\left[(z^2 + z)^{1/2} - z\right] \tag{30}$$

possesses an expansion of the type

$$f''(z) = \cfrac{1}{a_1 + \cfrac{1}{a_2 z + \cdots}}.$$

By adding together (29) and (30) we get the function which corresponds to (28). It is

$$f(z) = (z + \tfrac{1}{2})(z^2 + z)^{1/2} - z^2 - z + \tfrac{1}{8}. \tag{31}$$

Now, by comparing this with (27), we find that the two differ by only a constant, which is what we would expect in the light of Theorem 14.

We could also have proceeded as follows: Since $\Phi(x)$, as given by (28), decreases steadily up to $x = 1/2$, when it reaches the value $-1/(24\pi)$, and then increases, we may choose

$$\begin{cases} \Phi' = 0, & 0 < x < \tfrac{1}{2}, \\ \Phi' = \dfrac{1}{24\pi} - \dfrac{1}{3\pi}(x - x^2)^{3/2}, & \tfrac{1}{2} < x < 1, \\ \Phi' = \dfrac{1}{24\pi}, & 1 < x\,; \end{cases}$$

$$(32) \qquad \begin{cases} \Phi_1'' = -\dfrac{1}{3\pi}(x - x^2)^{3/2}, & 0 < x < \tfrac{1}{2}, \\ \Phi_1'' = -\dfrac{1}{24\pi}, & \tfrac{1}{2} < x. \end{cases}$$

In other words,

$$\frac{d\Phi'}{dx} = \frac{1}{\pi}\left(x - \frac{1}{2}\right)(x - x^2)^{1/2}, \quad \tfrac{1}{2} < x < 1,$$

and

$$\frac{d\Phi_1''}{dx} = -\frac{1}{\pi}\left(\frac{1}{2} - x\right)(x - x^2)^{1/2}, \quad 0 < x < \tfrac{1}{2},$$

the values of the derivatives being zero outside the indicated ranges. Then by direct computation we find for $f'(z)$

$$f'(z) = \frac{1}{4\pi(2z+1)} - \frac{1}{16} - \frac{z^2+z}{2} + \frac{2z+1}{4}(z^2+z)^{1/2} - \frac{1}{\pi}\frac{z^2+z}{2z+1}\left[F\left(1,1,\frac{1}{2}\,;\frac{1}{(2z+1)^2}\right) - 1\right],$$

where $F(\alpha, \beta, \gamma; x)$ stands for the hypergeometric function

$$F(\alpha, \beta, \gamma\,; x) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha\cdot\alpha+1\cdot\beta\cdot\beta+1}{1\cdot 2\cdot\gamma\cdot\gamma+1}x^2 + \cdots.$$

Similarly, we set $d\Phi''/dx = -(1/x)(d\Phi_1''/dx)$, compute $f''(z)$, and then make $f_1''(z) = zf''(z)$. The result is

$$f_1''(z) = \frac{z}{2\pi(2z+1)} - \frac{z^2+z}{2} + \frac{2z+1}{4}(z^2+z)^{1/2} + \frac{1}{\pi}\frac{z^2+z}{2z+1}\left[F\left(1,1,\frac{1}{2}\,;\frac{1}{(2z+1)^2}\right) - 1\right].$$

Adding f' and f_1'' together we get

$$(33)\quad f(z) = \left(\frac{1}{4\pi} - \frac{1}{16}\right) - z^2 - z + \left(z + \frac{1}{2}\right)(z^2 + z)^{1/2}.$$

This result again differs from (27) only to the extent of an additive constant. This time, however, the constant is smaller than before, being 0.017 instead of 0.125.

This value $(4-\pi)/(16\pi)$ is, in fact, the smallest difference which can exist between the sum of two fractions of the form

$$(34)\quad \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cdots}}} + \cfrac{1}{b_1 + \cfrac{1}{b_2 z + \cfrac{1}{b_3 + \cdots}}}$$

and the function (27), as we can readily show by the following argument.

It is evident that, at $z = \infty$, such a pair of fractions takes the value $1/b_1$. However, it is known (Theorem 2) that $1/b_1$ is equal to $\Phi''(\infty)$. Hence the fraction will have its least value at ∞ when $\Phi''(\infty)$ is smallest. We see at once from the relation between Φ'' and Φ_1'' that $\Phi''(\infty)$ will be smallest when $\Phi_1''(x)$ is given *for every* x the smallest value which is consistent with the requirement that Φ' shall nowhere decrease. This, however, is obviously the function $\Phi_1''(x)$ which we have adopted in (32). From it we find by direct computation that $\Phi''(\infty) = (4+\pi)/(16\pi)$. This, then, is the smallest value which $f(\infty)$ can have. We easily see from (27), however, that $f^*(\infty) = 1/8$. Hence the smallest *difference* that can exist between the two is $(4-\pi)/(16\pi)$, which is, indeed, the constant term in (33).

11. *The Final Process for Obtaining a Usable Expansion.* In the discussions of §§7–10 we have tacitly assumed that an expansion of the form (34) is satisfactory for purposes of design; and so it is if z is synonymous with our frequency variable p, for as we have seen in §1 the two fractions then correspond to constructible networks. But when the func-

tion $f^*(z)$ has been identified with $(cp+d)F^*(p)/(ap+b)$ or with $(\gamma p+\delta)F^*(p)/(\alpha p+\beta)$ by use of the transformation (13), the situation is not quite so simple, and something still remains to be accomplished.

To see why this is true, let us abbreviate the expressions $(ap+b)/(cp+d)$ and $(\alpha p+\beta)/(\gamma p+\delta)$ to q_1 and q_2, so that $z=q_1q_2$. Then from (34) we may get a pair of fractions representing $F(p)=q_1f(z)$. The terms of one of these are of the form aq_1 and aq_2 and are therefore constructible. But in the other, terms of the form $bq_1^2q_2$ appear, and these cannot be interpreted in electrical terms.† Hence the result, in exactly this form, is useless.

This difficulty can be overcome by separating $F^*(p)$ itself into two components each of which generates an expansion of usable form, instead of dealing with the transformed function $f^*(z)$.

In order to make our study somewhat more readable, we shall think entirely in terms of the transformation (15) and its accompanying Fig. 7. This is indeed quite general, for all distributions of singularities which can be enclosed by the boundary of (17) can also be enclosed by (15), and, as we have already observed, (16) can be obtained from (15) by a simple inversion on p. Then we have

$$(35) \qquad q_1 = \frac{rp}{(\lambda+1)p+s}, \qquad q_2 = \frac{rp}{(\lambda-1)p+s}.$$

We shall also assume $F^*(p)$ to be broken up into the sum of two components $G^*(p)$ and $H^*(p)$ by a procedure which will become clear as we proceed, and shall indicate the three functions of z into which they transform by $f^*(z)$, $g^*(z)$ and $h^*(z)$. That is, $f^*(z)\equiv F^*(p)$, $\cdots$.

We shall be interested in the values of all these various functions along the upper and lower edges of the cut in the

† We assume that $f^*(z)$ has been identified with $F^*(p)/q_1$. If it has been identified with $F^*(p)/q_2$ the fractions are slightly different, but still unsatisfactory.

z-plane. In the case of q_1 and q_2 we readily find them to be

$$(36) \qquad \begin{cases} rq_1 = x \pm i(r^2x - x^2)^{1/2}, \\ rq_2 = -x \pm i\,(r^2x - x^2)^{1/2}, \end{cases}$$

where the upper and lower signs apply to the upper and lower borders of the cut, respectively. About the values of f^*, g^* and h^* we can say nothing definite, but we introduce the notation

$$(37) \qquad \begin{cases} f^*(-x) = u_f \pm iv_f, \\ g^*(-x) = u_g \pm iv_g, \\ h^*(-x) = u_h \pm iv_h. \end{cases}$$

Then in terms of this notation, we readily find from (23) that the Φ's generated by $F^*(p)$, $G^*(p)/q_2$ and $H^*(p)/q_1$, are, respectively

$$(38) \qquad \begin{cases} \pi\dfrac{d\Phi_f(x)}{dx} = -v_f, \\ r\pi\dfrac{d\Phi_g(x)}{dx} = u_g\left(\dfrac{r^2}{x} - 1\right)^{1/2} + v_g, \\ r\pi\dfrac{d\Phi_h(x)}{dx} = u_h\left(\dfrac{r^2}{x} - 1\right)^{1/2} - v_h. \end{cases}$$

Now, what we purpose to show is, that so long as $F^*(p)$ is bounded along the entire boundary of the transformation,† $G^*(p)$ and $H^*(p)$ can always be so chosen that the left hand members of the second and third of equations (38) are everywhere positive. $\Phi_g(x)$ and $\Phi_h(x)$ then satisfy the conditions of Theorem 5. From them we can therefore obtain convergent Stieltjes fractions in the variable $z = q_1q_2$; whence, calling these fractions $G(p)/q_2$ and $H(p)/q_1$, respectively, we have

† This condition is sufficient, but not necessary, as we shall see in §12.

$$(39)\qquad \begin{cases} G(p) = \dfrac{1}{a_1q_1 + \dfrac{1}{a_2q_2 + \dfrac{1}{a_3q_1 + \cdots}}}, \\ H(p) = \dfrac{1}{b_1q_2 + \dfrac{1}{b_2q_1 + \dfrac{1}{b_3q_2 + \cdots}}}, \end{cases}$$

both of which are constructible.

To show that this is so, we start with the first of equations (38) and set $v_g' = v_f$ wherever v_f is positive, and $v_g' = 0$ elsewhere. Similarly we put $v_h' = v_f$ wherever v_f is negative, and zero elsewhere.† Thus we have

$$(40)\qquad v_g' + v_h' = v_f.$$

Next we define two new functions of z by the formulas

$$(41)\qquad \begin{cases} g'(z) = -\dfrac{1}{\pi}\displaystyle\int_0^{r^2} \frac{v_g'\,dx}{z+x}, \\ h'(z) = -\dfrac{1}{\pi}\displaystyle\int_0^{r^2} \frac{v_h'\,dx}{z+x}. \end{cases}$$

From (5), (40) and the first of equations (38) we see that $f'(z) = g'(z) + h'(z)$ is either identical with $f^*(z)$, or else differs from it only to the extent of a function $\delta(z)$ the Φ of which is everywhere zero. Further, by an argument essentially similar to that in §10 we may readily show that $g'(z)$ and $h'(z)$ are regular analytic functions as we approach the negative real axis from above and below, except perhaps at 0 and $-r^2$, and are bounded at these points. By analogy with (37) we shall call their values along the cut

$$(42)\qquad \begin{cases} g'(-x) = u_g' \pm iv_g', \\ h'(-x) = u_h' \pm iv_h', \end{cases}$$

† Any other method of division for which v_g' is everywhere positive and v_h' everywhere negative is satisfactory.

the v_g' being by definition everywhere *positive*, the v_h' everywhere *negative*, while u_g' and u_h' are only known to be *bounded*.

Finally, we take the arbitrary step of defining $g^*(z)$ and $h^*(z)$—and therefore also the $G^*(p)$ and $H^*(p)$ into which $F^*(p)$ is divided—by the equations

$$g^*(z) = g'(z) - M\left(\frac{r^2}{z} + 1\right)^{1/2},$$

$$h^*(z) = h'(z) + M\left(\frac{r^2}{z} + 1\right)^{1/2} + \delta(z),$$

M being a positive constant the value of which will be assigned later. Then we see that along the cut

$$g^*(-x) = u_g' \pm iv_g' \pm iM\left(\frac{r^2}{x} - 1\right)^{1/2},$$

$$h^*(-x) = u_h' \pm iv_h' \pm iM\left(\frac{r^2}{x} - 1\right)^{1/2};$$

and upon comparing these values with (37) we can identify u_g, v_g, u_h and v_h. Substituting the values thus found in the second and third of equations (38), we arrive at

$$(43) \qquad \begin{cases} r\pi \dfrac{d\Phi_g(x)}{dx} = (u_g' + M)\left(\dfrac{r^2}{x} - 1\right)^{1/2} + v_g', \\ r\pi \dfrac{d\Phi_h(x)}{dx} = (u_h' + M)\left(\dfrac{r^2}{x} - 1\right)^{1/2} - v_h'. \end{cases}$$

Now, since u_g' and u_h' are bounded, M may be taken so large that the first terms are positive, while the terms v_g' and $-v_h'$ are positive by definition. Hence both $\Phi_g(x)$ and $\Phi_h(x)$ are monotonic increasing and lead to Stieltjes fractions in z.

Thus we are assured that expansions of the form (39) really exist, where $G(p)$ and $H(p)$ are related to $F^*(p)$ by the equation

$$F^*(p) = G(p) + H(p) + \Delta(p),$$

$\Delta(p)$ being some function which generates a Φ that is everywhere zero. Hence we may state the following theorem.

THEOREM 15. *If $F^*(p)$ satisfies the conditions of Theorem 8 and is finite at the origin, it can be expressed as the sum of three functions $G(p)$, $H(p)$ and $\Delta(p)$, the first two of which possess constructible expansions of the form* (39), *while the third generates a Φ which is everywhere zero.*

We have thus avoided the difficulty to which attention was called at the beginning of the section.

12. *An Example.* We can make the material of §11 somewhat more intelligible by means of an illustration.

Let us consider the function†

$$F^*(p) = \left(p^2 + p + \frac{1}{2}\right) \log \frac{p+1}{p}.$$

By using (15) in the form

$$z = \frac{p^2}{(p+1)^2 - p^2}, \tag{44}$$

(which means, of course, that $r = s = \lambda = 1$), we readily convert $F^*(p)$ into

$$f^*(z) = \left(z + \frac{1}{2}\right)\left(z + \frac{1}{2} + (z^2 + z)^{1/2}\right) \log \frac{z+1}{z}.$$

Along the negative real axis this takes the value

$$\begin{aligned} f^*(-x) = {} & \left(x - \frac{1}{2}\right)^2 \log \frac{1-x}{x} + \pi\left(\frac{1}{2} - x\right)(x - x^2)^{1/2} \\ & \pm i\left[\left(\frac{1}{2} - x\right)(x - x^2)^{1/2} \log \frac{1-x}{x} - \pi\left(x - \frac{1}{2}\right)^2\right] \\ & \equiv u_f \pm i v_f. \end{aligned}$$

† This function has a pole at ∞, and is not bounded at the points 0 and -1. Thus it violates the conditions laid down in Theorem 15. The choice of such a function is deliberate, as it serves to emphasize that the conditons, while sufficient, are not always necessary.

This is the analog of the first of equations (37).

We now define

$$(45)\qquad \begin{cases} v_g' = \left(\frac{1}{2} - x\right)(x - x^2)^{1/2} \log \frac{1-x}{x}, \\ v_h' = -\pi\left(x - \frac{1}{2}\right)^2, \end{cases}$$

this definition serving the purpose of making v_g' everywhere positive and v_h' everywhere negative. Then we find from (41)

$$g'(z) = \left(z + \frac{1}{2}\right)\left(-1 + (z^2 + z)^{1/2} \log \frac{z+1}{z}\right),$$

$$h'(z) = \left(z + \frac{1}{2}\right)\left[-1 + \left(z + \frac{1}{2}\right) \log \frac{z+1}{z}\right].$$

We must next find the values (42) which these functions take along the negative real axis. Of course the imaginary components v_g' and v_h' are given by (45). The real components are readily found to be

$$(46)\qquad \begin{cases} u_g' = \left(\frac{1}{2} - x\right)\left[(-1 + \pi(x - x^2)^{1/2}\right], \\ u_h' = \left(\frac{1}{2} - x\right)\left[-1 + \left(\frac{1}{2} - x\right) \log \frac{1-x}{x}\right]. \end{cases}$$

Substituting (45) and (46) in (43), and remembering that $r = 1$, we obtain

$$\frac{d\Phi_g(x)}{dx} = \pi\left(\frac{1}{2} - x\right)(1 - x) + \left[M - \left(\frac{1}{2} - x\right) + x\left(\frac{1}{2} - x\right) \log \frac{1-x}{x}\right]\left(\frac{1-x}{x}\right)^{1/2};$$

$$\frac{d\Phi_h(x)}{dx} = \pi\left(\frac{1}{2} - x\right)^2 + \left[M - \left(\frac{1}{2} - x\right) + \left(\frac{1}{2} - x\right)^2 \log \frac{1-x}{x}\right]\left(\frac{1-x}{x}\right)^{1/2}.$$

Our problem now is to choose M so that these Φ's will be monotonic non-decreasing in the interval $0 \leqq x \leqq 1$. Inspection shows that this can be done by setting $M = 1/2$.

Using this value of M we readily find from (5) that the corresponding functions, which we will call $g(z)/q_2$ and $h(z)/q_1$, are

$$\frac{g(z)}{q_2} = \left(z + \frac{1}{2}\right)\left(z + \frac{1}{2} - (z^2 + z)^{1/2}\right) \log \frac{z+1}{z} + z - (z^2 + z)^{1/2},$$

$$\frac{h(z)}{q_1} = \left(z + \frac{1}{2}\right)^2 \left[1 + \left(\frac{z+1}{z}\right)^{1/2}\right] \log \frac{z+1}{z} - z - 1 - (z^2 + z)^{1/2}.$$

Both of these functions, then, have Stieltjes expansions.

But by (44) and (35),

$$z = \frac{p^2}{2p+1}, \quad q_1 = \frac{p}{2p+1}, \quad q_2 = p.$$

Hence

$$G(p) \equiv g(z) = \frac{2p^2 + 2p}{(2p+1)^2}\left(p^2 + p + \frac{1}{2}\right) \log \frac{p+1}{p} - \frac{p^2}{2p+1}$$

and

$$H(p) \equiv h(z) = \frac{2p^2 + 2p + 1}{(2p+1)^2}\left(p^2 + p + \frac{1}{2}\right) \log \frac{p+1}{p} - \frac{p^2 + p}{2p+1}$$

must have expansions in the form (39).

We see at once that

$$F^*(p) = G(p) + H(p) + p,$$

the extra term p being an entire function which generates a zero $\Phi(x)$. It occurs because $F^*(p)$ has a pole at ∞.

This result is an entirely satisfactory one from the standpoint of design, for not only $G(p)$ and $H(p)$, but $\Delta(p) \equiv p$ as well, are constructible, the resultant structure consisting of an inductance and two ladder networks in series.

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A CORRECTION

BY E. T. BELL

Dr. M. A. Basoco has kindly pointed out to me that the isomorphism established in my paper in the last issue of this Bulletin,* is not, as there stated, partial, but complete. In the second sentence of §3, page 323, the word *not* should be deleted. Obviously the relation stated is identical with (B). On page 324 (8), the words *but not* should be replaced by the word *and*. The error arose from replacing a certain intermediary function of my first draft by $\lambda(z)$; the discarded function did not satisfy (B), but it failed to give a satisfactory analogy in some other respects.

As the matter now stands, it is clear that *more* is proved than was stated in my paper: *my revised Lucas functions are identical with the stated Weierstrassian normal forms having the given invariants.* Complete isomorphism *is* possible, provided we attend in Halphen's theorem to the degenerate case of one period infinite.

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* *A partial isomorphism between the functions of Lucas and Weierstrass*, this Bulletin, vol. 35 (1929), pp. 321–325.

THE APPROXIMATION OF HARMONIC FUNCTIONS BY HARMONIC POLYNOMIALS AND BY HARMONIC RATIONAL FUNCTIONS*

BY J. L. WALSH

1. *Introduction.* The following theorem of Weierstrass is classical.

Let the function $f(\theta)$ be continuous for all values of the argument and periodic with period 2π. Then if an arbitrary positive ϵ be given, there exists a trigonometric polynomial which differs from $f(\theta)$ at most by ϵ; that is, the inequality

$$(1) \qquad \left| f(\theta) - \sum_{k=0}^{n} (a_k \cos k\theta + b_k \sin k\theta) \right| \leqq \epsilon$$

holds for all values of θ.

An equivalent statement of the conclusion of this theorem is that *$f(\theta)$ can be expanded in a series of the form*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} (a_{nk} \cos k\theta + b_{nk} \sin k\theta),$$

which converges uniformly for all values of θ. This is of course a general fact, that if a given function can be uniformly approximated as closely as desired by a linear combination of other functions, then that function can be expanded in a uniformly convergent series of which each term is a linear combination of those other functions, and conversely. This fact is easy to prove and will be frequently used in the sequel.

If in the (x, y)-plane we introduce polar coordinates (r, θ) and consider the function $f(\theta)$ defined on the unit circle C, Weierstrass's theorem refers to the approximation of a function $f(\theta)$ continuous on C by trigonometric polynomials, or what is the same thing, by polynomials of the form

* An address delivered by invitation of the program committee at the meeting of the Society in New York, February 23, 1929.

$$\sum_{k=0}^{n} r^k(a_k \cos k\theta + b_k \sin k\theta). \tag{2}$$

A polynomial of the form (2) is of course harmonic, for the expressions $r^k \cos k\theta$ and $r^k \sin k\theta$ are respectively the real parts of the analytic functions z^k, $-iz^k$, and can be written as harmonic polynomials in x and y.

If we interpret Weierstrass's theorem as yielding a uniform *expansion* rather than uniform approximation of $f(\theta)$, we have a sequence $\{p_n(r, \theta)\}$ of functions each of type (2) which converges uniformly to the function $f(\theta)$ on C and likewise converges uniformly in the *closed* interior of C. For a necessary and sufficient condition for uniform convergence on C is that if an arbitrary ϵ be given, there exist M so that

$$| p_m(r,\theta) - p_n(r,\theta) | \leqq \epsilon, \quad m,n > M,$$

holds on C. But this inequality holding for $r=1$ holds also for $r \leqq 1$, since the functions $p_n(r, \theta)$ are harmonic for $r \leqq 1$ and a harmonic function has its maximum and minimum values on the boundary of the region considered. The limit $f(r, \theta)$ of this sequence $\{p_n(r, \theta)\}$ is thus harmonic* interior to C, continuous in the corresponding closed region, and on C coincides with the given continuous function $f(\theta)$. That is, by Weierstrass's theorem we have established the existence of the function $f(r, \theta)$, the solution of the Dirichlet problem for the region interior to C and for arbitrary continuous boundary values $f(\theta)$; the solution of the Dirichlet problem is known to be unique.

The problem of the uniform approximation to $f(\theta)$ *on* C by harmonic polynomials is precisely equivalent to the problem of the uniform approximation to $f(r, \theta)$ *in the closed interior of* C, for a function harmonic in a region has no maximum or minimum interior to that region. Inequality (1) is equivalent to the inequality

* A function $u(x, y)$ is harmonic in a region if it is continuous there, together with its first and second partial derivatives, and if $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$. A function is harmonic at a point if it is harmonic throughout a neighborhood of that point.

$$\left| f(r,\theta) - \sum_{k=0}^{n} r^k (a_k \cos k\theta + b_k \sin k\theta) \right| \leqq \epsilon, \qquad r \leqq 1;$$

each of these inequalities implies the other.

The discussion just given suggests the general problem of studying the approximation to a given continuous function on an arbitrary point set C by means of harmonic polynomials or more generally by harmonic rational functions, particularly in the more interesting cases that the given function is harmonic in the interior points of C, or that C is itself the boundary of a region. The present paper is devoted to results on this general problem, chiefly a *report* so far as concerns approximation by polynomials, but a detailed exposition of results on approximation by more general rational functions, for these results are in the main new. As in the special case already considered to some extent, namely the theory of approximation to functions of a single real variable by trigonometric polynomials, there are particularly three special problems to be taken up here: first, the general problem of the possibility of approximation, corresponding to Weierstrass's theorem as already mentioned; second, the problem of approximation by *special types* of polynomials corresponding for example to expansion in Fourier's series; and third, the study of the *degree* of approximation, the study of the relation between the continuity properties of the functions approximated and the least maximum error for approximation by a polynomial of degree n. We shall examine these problems in some detail in the order given, first the problems for approximation by polynomials and later the corresponding problems for approximation by more general rational functions.

In the theory of functions of the complex variable $z=x+iy$, similar problems arise in connection with the approximating on a given point set of a given continuous or analytic function by means of polynomials in z or rational functions of z. This analogous theory frequently suggests results in the present theory of harmonic functions, and

may even furnish methods of proving those results; but the connection between the two theories is not always simple, and some situations in each theory have no analogs in the other. The two theories both have interesting applications to and connections with harmonic and analytic continuation, conformal mapping, the study of the Dirichlet problem, and topology.

2. *General Approximation by Harmonic Polynomials.* The fundamental theorem in the approximation of analytic functions by polynomials is due to Runge [1]*:

Let $f(z)$ be a function of z, analytic in a simply-connected region† C of the z-plane which does not contain the point at infinity in its interior. Then $f(z)$ can be developed in C in a series of polynomials in z, which converges uniformly on any closed point set interior to C.

We shall use Runge's theorem to prove the following analog for harmonic functions.‡

Let $u(x, y)$ be a function of (x, y), harmonic in a simply-connected region C of the (x, y)-plane which does not contain the point at infinity in its interior. Then $u(x, y)$ can be developed in C in a series of harmonic polynomials in (x, y), which converges uniformly on any closed point set interior to C.

Let $v(x, y)$ be a function conjugate to $u(x, y)$ in C; the function $v(x, y)$ is single-valued in C if one considers only an element of $v(x, y)$ and its harmonic extensions along curves interior to C. Runge's theorem as applied to the function $f(z) = u(x, y) + iv(x, y)$ yields a development of $f(z)$ in polynomials in z. The series whose terms are the real parts of the respective terms of that development represents the function $u(x, y)$ and has the required properties with respect to convergence.

* The numbers in square brackets refer to the works of the authors indicated, in the bibliography at the close of this paper.

† A region is a connected set of interior points. A region plus its boundary points forms the corresponding *closed* region.

‡ Compare Walsh [1, p. 198]. Bergmann [1] uses similar reasoning, but does not bring out clearly the uniformity of the convergence of the series of harmonic polynomials.

The essential content for our present purposes of the theorem just proved may also be stated as follows.

If the function $u(x, y)$ is harmonic in a closed Jordan region, then $u(x, y)$ can be developed in that closed region in a uniformly convergent series of harmonic polynomials in (x, y).

The present writer has proved, however, results much more general than this [1], regarding the uniform development of arbitrary harmonic functions. We state a number of these results, simply mentioning at this time the fact that the proof is based on Lebesgue's important work [1] on harmonic functions. We shall go into more detail regarding methods in §5, when we take up the more general problem of development in terms of harmonic rational functions.

Let C be an arbitrary limited simply-connected region of the (x,y)-plane. A necessary and sufficient condition that an arbitrary function harmonic in C, continuous in the corresponding closed region, be uniformly developable in the closed region by harmonic polynomials in (x, y), is that the boundary of C be also the boundary of an infinite region.

Let C be an arbitrary limited closed point set. A necessary and sufficient condition that an arbitrary function continuous on C be uniformly developable on C by harmonic polynomials in (x, y), is that C be the boundary of an infinite region.

An arbitrary limited closed point set C divides the plane in general into a number of regions; in particular one of these, D, is infinite. A continuous function $u(x, y)$ defined on C is in particular defined on the boundary B of D, and on this boundary can, by the theorem just stated, be uniformly developed in harmonic polynomials. This expansion of $u(x, y)$ on B converges uniformly on B, hence uniformly on the entire point set E which is complementary (with respect to the entire plane) to D, and therefore defines a function $U(x, y)$ on the entire set E. The set C is a subset of E. The functions $u(x, y)$ and $U(x, y)$ coincide on B. *A necessary and sufficient condition that $u(x, y)$ be uniformly developable on C in harmonic polynomials is that $u(x, y)$ and $U(x, y)$ should coincide on C.*

These theorems obviously include Weierstrass's theorem on the approximation to an arbitrary continuous function by trigonometric polynomials. These theorems include also Weierstrass's classical theorem that an arbitrary function continuous on a closed interval of the axis of reals can be uniformly approximated on that closed interval as closely as desired by a polynomial in the real variable. For if the point set C is chosen as such an interval, the only harmonic polynomials $1, r\cos\theta, r\sin\theta, \cdots, r^n\cos n\theta, r^n\sin n\theta, \cdots$ not vanishing identically on C can be written on C in the form $1, x, x^2, \cdots$, so that on C a harmonic polynomial is a polynomial in x.

With these results, the most important questions regarding the expansion of arbitrary functions on *limited* point sets are answered. The corresponding questions for unlimited point sets have apparently not been treated previously. On some unlimited point sets C only the polynomial 1 of the set $\{r^n\cos n\theta, r^n\sin n\theta\}$ is uniformly bounded, so the only functions continuous on C which can be uniformly approximated on C by harmonic polynomials are constants. On certain other unlimited point sets C only a finite number of the harmonic polynomials $\{r^n\cos n\theta, r^n\sin n\theta\}$ are continuous, and a necessary and sufficient condition that a function continuous on such a point set C be uniformly developable on C by harmonic polynomials is that the function should be a linear combination of those particular harmonic polynomials. On other unlimited point sets C an infinity of the harmonic polynomials $\{r^n\cos n\theta, r^n\sin n\theta\}$ are continuous, so necessary and sufficient conditions for the expansion of arbitrary functions on such point sets C are not obvious, nor can they be obtained without modification of the methods hitherto employed.

It is illuminating to contrast the present problem with the problem of approximating given analytic or continuous functions by means of polynomials in the complex variable. It is not yet known what is the most general region C such that an

arbitrary function analytic interior to C and continuous in the corresponding closed region can be approximated uniformly in the closed region as closely as desired by polynomials in the complex variable. It can be shown by an example [Walsh, 1, p. 203] however, that this most general region C is not the most general region such that an arbitrary function harmonic interior to C and continuous in the corresponding closed region can be approximated uniformly in the closed region as closely as desired by harmonic polynomials. *Unlimited* point sets C are easy to treat for approximation by polynomials in the complex variable. Every such polynomial not a constant becomes infinite on C, so the only continuous functions which can be uniformly approximated on C are constants, and every such function can be so approximated.

3. *Harmonic Polynomials Belonging to a Region.* It is natural to inquire whether more precise results than those just considered are obtainable regarding *expansions in a given region in terms of a particular set of harmonic polynomials belonging to that region.* For instance, if the region is chosen as the interior of the unit circle C, then an arbitrary function $f(x, y)$ harmonic in the closed interior of C can be expanded on the circumference in a series of Fourier:

$$(3)\quad f(x,y) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta),$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f \cos k\theta \, d\theta, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f \sin k\theta \, d\theta,$$

where the integrals are computed over C. The series converges uniformly on C. The series

$$(4)\qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k(a_k \cos k\theta + b_k \sin k\theta)$$

is a series of harmonic polynomials which converges uni-

formly on C and hence converges uniformly throughout the closed interior of C. The sum of the series (4) coincides with $f(x, y)$ on C and hence throughout the interior of C as well.

If we start not with the unit circle but a more general Jordan curve C as the boundary of our region, can there be found harmonic polynomials $\{p_k(x, y)\}$ such that an arbitrary function harmonic on and within C can be expanded in a series

$$\sum_{k=0}^{\infty} a_k p_k(x, y) \tag{5}$$

which converges uniformly in the closed interior of C? The answer here is affirmative, if the curve C is analytic. In fact, Faber [1], Fejér [1], Szegö [1], Carleman [1], Bergmann [1], Bochner [1], and others have studied polynomials $\{s_k(z)\}$ in the complex variable z belonging to such a region and have shown that an arbitrary function $f(z)$ analytic in the closed region can be expanded in a series

$$\sum_{k=0}^{\infty} c_k s_k(z)$$

which converges uniformly in the closed region. A suitable set of harmonic polynomials of the kind we desire can be found by separating $s_k(z)$ into its real and pure imaginary parts. Thus, let the function $u(x, y)$ be given harmonic in the closed interior of C; there exists a function $v(x, y)$ conjugate to it, likewise harmonic in this closed region. Then by the properties of the polynomials $s_k(z)$, there exists an expansion

$$u(x,y) + iv(x,y) = \sum_{k=0}^{\infty} (c_k' + ic_k'')(s_k'(x,y) + is_k''(x,y)),$$

where $c_k = c_k' + ic_k''$, $s_k(z) = s_k'(x, y) + s_k''(x, y)$. From this we derive the series

$$u(x,y) = \sum_{k=0}^{\infty} [c_k' s_k'(x,y) - c_k'' s_k''(x,y)], \tag{6}$$

so we have an expansion of type (5)* by setting

$$p_{2k}(x,y) = s_k'(x,y), \qquad p_{2k+1}(x,y) = s_k''(x,y).$$

In particular if the derivatives of the polynomials $s_k(z)$ are orthogonal with respect to the area interior to C, the formulas for the coefficients in (6) are remarkably simple in form, as Bergmann has shown [1].

There are, however, various other ways of determining sets of harmonic polynomials belonging to a given region. Let us consider the plane of the auxiliary variables (x', y'), and map conformally the interior of C onto the interior of the unit circle Γ of the (x', y')-plane. A function harmonic in the closed interior of C corresponds to a function harmonic in the closed interior of Γ, and can be uniformly expanded in the closed interior of Γ in a series (4) of harmonic polynomials in (x', y'). The harmonic polynomials $r^n \cos n\theta$, $r^n \sin n\theta$ in the (x', y')-plane correspond in the (x, y)-plane to functions harmonic in the closed interior of C; moreover these functions in the (x, y)-plane can be replaced by harmonic *polynomials* in (x, y) which differ only slightly from them, without essentially altering the convergence properties of series expansions of arbitrary functions in terms of them. More explicitly, we state the following theorem [Walsh, 2].

Let C be a simple finite analytic curve in the (x, y)-plane. Then there exist harmonic polynomials $\{p_k(x, y)\}$ such that if $f(x, y)$ is defined and continuous on C and on C is of bounded variation, then $f(x, y)$ can be developed into a series

$$f(x,y) = \sum_{k=0}^{\infty} a_k p_k(x,y), \tag{7}$$

which converges uniformly in the closed interior of C. Series (7) thus represents a function harmonic interior to C, continuous in the corresponding closed region, and having the value $f(x, y)$ on C. There exist continuous functions $\{q_k(x, y)\}$ on C

* Removal of brackets in series (6) can be justified in the cases to which reference has been made.

with which the polynomials $\{p_k(x, y)\}$ *form a biorthogonal set*:

$$\int_C p_k(x,y)q_m(x,y)ds = \begin{cases} 0, & \text{if } k \neq m, \\ 1, & \text{if } k = m. \end{cases}$$

The coefficients of (7) *are given by the formulas*

$$a_k = \int_C f(x,y)q_k(x,y)ds \; ; \tag{8}$$

the functions $q_k(x, y)$ *depend on C but not on* $f(x, y)$.

If the function $f(x, y)$ *is known merely to be continuous on C, then the series* (7), *where the* a_k *are given by* (8), *converges throughout the interior of C, uniformly on any closed point set interior to C; if summed by the method of Cesàro this series converges uniformly on and within C and thus represents a solution of the Dirichlet problem for the region interior to C and the boundary values* $f(x, y)$.

If the function $f(x, y)$ *is of bounded variation on C, but not necessarily continuous, then the series* (7), *where the* a_k *are given by* (8), *converges at every point of the closed region, uniformly on any closed point set interior to C. The function represented is bounded in the closed interior of C and approaches the boundary values* $f(x,y)$ *continuously at every point of continuity of* $f(x, y)$.

This theorem as stated applies only to an analytic curve C, but the writer has some further results, as yet unpublished, which apply to much more general curves.

Still another method of defining a set of harmonic polynomials belonging to a given region bounded by a rectifiable Jordan curve is that of orthogonalization. We begin with the harmonic polynomials

$$1, \; r\cos\theta, \; r\sin\theta, \; r^2\cos 2\theta, \; r^2\sin 2\theta, \; \cdots$$

and orthogonalize them with respect to the given curve C. This yields a set of polynomials $\{p_k(x, y)\}$ such that

$$\int_C p_k(x,y)p_m(x,y)ds = \begin{cases} 0, & k \neq m, \\ 1, & k = m. \end{cases}$$

An arbitrary function $f(x, y)$ defined on C can be expanded formally in terms of these polynomials:

$$(9) \quad f(x,y) \sim \sum_{k=0}^{\infty} a_k p_k(x,y), \quad a_k = \int_C f(x,y) p_k(x,y) ds.$$

This orthogonalization method was used by Szegö [1] in the corresponding case of polynomials in the complex variable. The results in the present case have been established by Merriman [1], who determines the asymptotic formulas for the polynomials $\{p_k(x, y)\}$, and shows, under suitable restrictions, that the series (9) converges interior to C or even uniformly in the closed interior of C. The sum of the series is of course the solution of the Dirichlet problem for the continuous boundary values $f(x, y)$.

It would be of interest to investigate the corresponding problems where the harmonic polynomials are orthogonalized not with respect to arc length on the boundary, but with respect to the area of the region interior to C [compare Bergmann 1, Carleman 1]. The former method has the theoretical advantage, however, of not requiring for expansion the knowledge of the values of the harmonic function except on the boundary of the region. Another general problem which deserves treatment is the detailed study of the convergence on C of expansions in terms of the special polynomials of Faber and others, either for the case of analytic or of harmonic functions, where the given function to be expanded is not known to be analytic or harmonic in the *closed* region.

4. *Degree of Approximation.* We have considered thus far the *possibility* of approximation by harmonic polynomials and the expansion in terms of a *particular set* of harmonic polynomials. We turn now to consideration of the *degree* of approximation, that is, the relation between the properties of the function approximated, with regard to existence of derivatives etc. on the one hand, and the asymptotic properties of the maximum error in the best approximation

by a harmonic polynomial of degree n,* on the other hand. A simple result [Walsh, 3] here is the analog and consequence of the corresponding result in approximation of analytic functions by means of polynomials in the complex variable.

Let C be an arbitrary closed Jordan region of the (x, y)-plane, and let $w=\phi(z)$, $z=x+iy$, be a function which maps conformally the exterior of C onto the exterior of the unit circle in the w-plane so that the points at infinity correspond to each other. Let C_R denote the curve $|\phi(z)|=R$, $R>1$, that is, the transform in the z-plane of the circle $|w|=R$.

A necessary and sufficient condition that an arbitrary function $u(x, y)$, defined in C, be harmonic in the (closed) region C is that there should exist harmonic polynomials $p_n(x, y)$ of degree n, $n=0, 1, 2, \cdots$, and numbers M, $R>1$, such that the inequalities

$$|u(x,y)-p_n(x,y)| \leqq \frac{M}{R^n}, \tag{10}$$

where M and R are independent of n and of (x, y), should be valid for every point (x, y) of C.

If the polynomials $p_n(x, y)$ are given so that (10) is satisfied for every (x, y) of C, the sequence $\{p_n(x, y)\}$ converges everywhere interior to C_R and uniformly on any closed point set interior to C_R, so the function $u(x, y)$ is harmonic throughout the interior of C_R.†

If $u(x, y)$ is given harmonic in the closed region interior to C_ρ, the polynomials $p_n(x, y)$ can be chosen to satisfy (10) with $R=\rho$, for (x, y) in C.

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* The polynomial $a_0x^n+a_1x^{n-1}y+\cdots+a_ny^n+b_0x^{n-1}+b_1x^{n-2}y+\cdots$ is considered to be of degree n, but if the term degree is used *in the restricted sense*, is of degree n if and only if at least one of the coefficients a_i is different from zero.

† Here and below we tacitly assume that if $u(x, y)$ is not originally supposed to be defined on the entire point set considered, then the definition in the new points is to be made by harmonic extension—or what amounts to the same thing—by means of the convergent series of harmonic polynomials.

This theorem seems to be the only one in the literature concerning degree of approximation by harmonic polynomials, except in the case that C itself is a circle, when approximation by harmonic polynomials reduces to approximation on the circumference by trigonometric polynomials, for which well known results have been obtained by Bernstein, Jackson, Montel, de la Vallée-Poussin, and others [see de la Vallée-Poussin, 1]. There is obviously occasion here for further investigation. If $u(x, y)$ is given harmonic within, continuous on and within the Jordan curve C, what is the relation between the maximum error for the best approximation of $u(x, y)$ on C by a harmonic polynomial of degree n on the one hand, and the continuity properties of the curve C and of the function $u(x, y)$ on C on the other hand? If $u(x, y)$ is given harmonic within, continuous on and within C_R, what is the relation between the maximum error for the best approximation on C by a harmonic polynomial of degree n on the one hand, and the continuity properties of the curve C and of the function $u(x, y)$ on the curve C_R on the other hand?

Let us stop for a moment to consider the Tchebycheff harmonic polynomial for the function $u(x, y)$ on a point set C, that is, the harmonic polynomial $p_n(x, y)$ of degree n for which the maximum $|u(x, y)-p_n(x, y)|$, (x, y) on C, is least. It is convenient here to refer to a general theorem due to Haar [1], which deals with approximation on a given point set C of a given function $u(x, y)$ by linear combinations of given functions $\{u_i(x, y)\}$; the coefficients a_i are to be determined so that the maximum

$$(10') \qquad |u(x, y) - a_1u_1(x, y) - \cdots - a_mu_m(x, y)|$$

is least for (x, y) on C. If C is closed and the u's are continuous, such a determination of the coefficients is possible [Haar, loc. cit.] and the corresponding linear combination of the $u_i(x, y)$ may be called a Tchebycheff polynomial of order m. Haar's theorem asserts—except for incidental restrictions on the continuity of the functions and the

closure of the point set—that *a necessary and sufficient condition for the uniqueness of the Tchebycheff polynomial of order m for approximation on C to an arbitrary function continuous on C, is that no function*

$$(10'') \qquad \begin{aligned} &A_1u_1(x,y)+\cdots+A_mu_m(x,y),\\ &|A_1|+|A_2|+\cdots+|A_m|\neq 0, \end{aligned}$$

shall vanish at more than $m-1$ points of C. It is immaterial whether one studies on the one hand the approximation in a closed Jordan region C by harmonic polynomials of a function harmonic interior to the region, continuous in the closed region, or on the other hand the approximation on the boundary of the region of the boundary values of the harmonic function. For the maximum in C of the function in (10′) must occur on the boundary of C. Fréchet has proved that the Tchebycheff trigonometric polynomial is unique for approximation to an arbitrary continuous function $f(\theta)$ with period 2π on the interval $0\leqq\theta\leqq 2\pi$. That is, from our present standpoint (compare §1), *the Tchebycheff harmonic polynomial is unique for approximation to a continuous function on a circumference C, or for approximation on and within a circle to a function continuous there, harmonic in the interior.* This statement is true, however, only with a restriction. The Tchebycheff polynomial which is a linear combination of the functions $1, r\cos\theta, r\sin\theta, \cdots, r^n\cos n\theta, r^n\sin n\theta$ is indeed unique, for an arbitrary linear combination (10″) of these functions vanishes on an algebraic curve of degree n, and this curve has at most $2n$ points in common with the circumference C.* In fact, this method proves that *the corresponding Tchebycheff harmonic polynomial is unique for approximation on an ellipse C to a function continuous on C, or for approximation on and within C of a function harmonic interior to C, continuous in the corresponding closed region.*

The Tchebycheff polynomial, which is a linear combination

* The circumference cannot be a branch of this curve, for the function (10″) cannot vanish everywhere on any (limited) Jordan curve.

of the functions $1, r\cos\theta, r\sin\theta, \cdots, r^{n-1}\cos(n-1)\theta, r^{n-1}\sin(n-1)\theta, r^n\cos n\theta$, is not necessarily unique either when C is a circle or an ellipse. Indeed, if $n=1$, the function (10″) is A_1+A_2x, which obviously for suitable choice of A_1 and A_2 vanishes at more than a single point of the circle or ellipse C.

It would be an interesting investigation, to determine what algebraic curves C have this property, that the Tchebycheff harmonic polynomial for approximation to an arbitrary function continuous on C is unique. This is a subject dealing with the real intersections of plane algebraic curves, in which a single point of intersection counts merely as a single point, in spite of singularities and multiple points (in the usual sense of the term) of either curve.

Whether or not the Tchebycheff harmonic polynomial $t_n(x, y)$ of degree n is unique, a Tchebycheff polynomial $t_n(x, y)$ of degree n exists, and we can derive certain properties of the sequence $\{t_n(x, y)\}$. In the notation of the previous theorem, we shall prove the following facts.

Let the function $u(x, y)$ be harmonic in the closed interior of C, and have at least one singularity on the curve C_ρ but no singularity interior to C_ρ. Then a sequence $\{t_n(x, y)\}$ of Tchebycheff harmonic polynomials for the function $u(x, y)$ considered in the closed interior of C (or what is essentially the same, considered on C itself), whether or not the Tchebycheff polynomial is unique, converges throughout the interior of C_ρ, and uniformly on any closed point set interior to C_ρ. The limit of the sequence throughout the interior of C_ρ is $u(x, y)$. The sequence can converge uniformly in no region $C_{\rho'}$ with $\rho' > \rho$.

The proof is immediate. There exists, by the theorem already stated, some sequence $\{p_n(x, y)\}$, where $p_n(x, y)$ is of degree n, such that we have

$$|u(x,y) - p_n(x,y)| \leqq \frac{M}{R^n}, \qquad (x,y) \text{ in } C, \tag{10}$$

provided merely that $R<\rho$. If (10) is valid for the poly-

nomial $p_n(x, y)$, it is also valid if $p_n(x, y)$ is replaced by $t_n(x, y)$:

$$| u(x,y) - t_n(x,y) | \leqq \frac{M}{R^n}, \qquad (x,y) \text{ in } C,$$

provided still that $R<\rho$. Again by virtue of the theorem stated, the sequence $\{t_n(x, y)\}$ must converge interior to every C_R for which $R<\rho$, hence throughout the interior of C_ρ, uniformly on any closed point set interior to C_ρ. In particular, if the function $u(x, y)$ has no singularity except at infinity, the sequence converges at every point of the plane, uniformly on any limited closed point set. In the general case, the sequence $\{t_n(x, y)\}$ cannot converge uniformly on any curve $C_{\rho'}$ for which $\rho'>\rho$, for then it would likewise converge uniformly in a region containing C_ρ, which has on it a singularity of the function $u(x, y)$.

The theorem just established is the analog of a theorem due to Faber [2, p. 105] for the case of approximation to an analytic function by polynomials in the complex variable. The proofs are, however, different, and the present proof applies without essential change in that other situation, even if C is a Jordan arc or a certain more general point set, instead of a Jordan region.

It will be noticed that the sequence $\{t_n(x, y)\}$ may converge for certain points (x, y) exterior to C_ρ. Let us suppose for instance that the region C is symmetric on the x-axis and that the function $u(x, y)$ satisfies the equation $u(x, y) = -u(x, -y)$. Then a given $t_n(x, y)$ may be replaced by another Tchebycheff polynomial $t_n'(x, y)$ of degree n which satisfies the equation $t_n'(x, y) = -t_n'(x, -y)$ and which approximates $u(x, y)$ on C as closely as does $t_n(x, y)$. If we have

$$| u(x,y) - t_n(x,y) | \leqq \epsilon, \qquad (x,y) \text{ on } C,$$

we have likewise by symmetry

$$| u(x, -y) - t_n(x, -y) | \leqq \epsilon, \qquad (x,y) \text{ on } C,$$

that is,

$$| u(x,y) + t_n(x,-y) | \leqq \epsilon, \qquad (x,y) \text{ on } C.$$

If we set

$$t_n'(x,y) = \frac{t_n(x,y) - t_n(x,-y)}{2},$$

the harmonic polynomial $t_n'(x, y)$ obviously satisfies the functional equation considered, and we have the inequality

$$| u(x,y) - t_n'(x,y) | \leqq \epsilon, \qquad (x,y) \text{ on } C;$$

that is to say, the polynomial $t_n'(x, y)$ is as good an approximation to $u(x, y)$ on C as is $t_n(x, y)$.* Interpret $t_n'(x, y)$ as a linear combination of the functions $r^n \cos n\theta, r^n \sin n\theta$, hence as a linear combination of the functions $r^n \sin n\theta$. The sequence $\{t_n'(x, y)\}$ converges for $\theta = 0$ for *all* values of r, no matter how small C may be or where the singularities of $u(x, y)$ may lie.

We have thus far restricted our entire discussion of the degree of approximation by harmonic polynomials to the consideration of point sets C which are regions. That is probably the simplest case; the study of the same problems where C is, let us say, a Jordan arc, is more complicated. Complication arises because a harmonic polynomial may vanish identically on C, so that convergence on C of a sequence of harmonic polynomials, even so that (10) is satisfied on C, does not imply convergence of the sequence for points not on C. Let us treat here in detail the simplest pos-

* We have essentially proved here that if the function $u(x, y)$ can be uniformly expanded by harmonic polynomials on a point set C which is symmetric in the x-axis and if we have $u(x, y) = -u(x, -y)$, then $u(x, y)$ can likewise be uniformly expanded on C by harmonic polynomials $p_n(x, y)$ which satisfy the equation $p_n(x, y) = -p_n(x, -y)$.

Another theorem of the same general nature easily proved by methods already used elsewhere for functions of a complex variable is that if an arbitrary function $u(x, y)$ can be uniformly expanded by harmonic polynomials on a point set C, then $u(x, y)$ is the uniform limit of a sequence of harmonic polynomials each of which is equal to $u(x, y)$ in n arbitrary preassigned points of C.

sible case, namely that C is an interval of the axis of reals; the result is also valid if C is any line segment.

Suppose that $u(x, y)$ is defined on C and that a set of polynomials $\{p_n(x, y)\}$ exists so that (10) holds on C for $R>1$. On C a harmonic polynomial of degree n is a polynomial in x of degree n. Inequality (10), valid for a polynomial $p_n(x, y)$ in x of degree n, implies the convergence of this sequence of polynomials *considered as polynomials in the complex variable* x throughout the interior of C_R (which is defined here as already indicated for a *region* C), uniformly on any closed point set interior to C_R, and hence represents interior to C_R an analytic function of the complex variable x. This is indeed a theorem due to Bernstein, and in the present case the curve C_R is a certain ellipse whose foci are the extremities of the interval C. It follows that *there exists a function* $U(x, y)$, *harmonic interior to* C_R, *coinciding on* C *with the given function* $u(x, y)$. There are two important differences between this result and the result established for the case that C is a Jordan region. First, we have not shown in the present case, nor is it necessarily true, that the original given sequence of harmonic polynomials $\{p_n(x, y)\}$ converges everywhere interior to C_R. Second, the function $U(x, y)$ is not uniquely determined by the requirements of being harmonic interior to C_R and coinciding on C with the given function $u(x, y)$; if any such function $U(x, y)$ is at hand, we may find another by adding to it an arbitrary function harmonic interior to C_R and vanishing on C.

Reciprocally, if it is desired to establish (10) when the function $u(x, y)$ is given harmonic on and within C_R, that can always be done if C is an arbitrary Jordan arc, or indeed a much more general point set. The result follows from the corresponding result for the development of analytic functions in terms of polynomials in the complex variable [compare Walsh, 3].

Before we leave the subject of approximation by harmonic polynomials and turn to harmonic rational functions, we mention another topic which seems not to have been treated

in the literature and yet which deserves to be investigated, namely, interpolation by means of harmonic polynomials. When does a harmonic polynomial of degree n exist which takes on preassigned values at $2n+1$ points? When is it unique? What of the asymptotic character of the polynomial as the number of points becomes infinite? What application is there to approximation and expansion, if these points are chosen on the boundary of a region? These questions have been answered in special cases, corresponding to trigonometric interpolation [see Faber, 3, Jackson, 1, de la Vallée-Poussin, 1], and also for the analogous problem of interpolation by polynomials in the complex variable [Fejér, 1]. The former case yields satisfactory results (compare §1) for interpolation by harmonic polynomials either on a circumference or on and within a circle, but for more general situations the questions seem still to be untouched.*

5. *General Approximation by Harmonic Rational Functions.* The results thus far established for approximation by harmonic polynomials have precise analogs for approximation of arbitrary harmonic functions by harmonic rational functions; these analogs will now be treated in the same order, and the proofs follow, in the main, the proofs for the simpler case. The results in the present case, however, are mostly new to the literature and must therefore be treated here in some detail. This newness explains the apparent lack of balance between the treatment of the two cases of approximation, by harmonic polynomials and by more general harmonic rational functions.

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* These results of Fejér do yield simple theorems immediately. Thus we may make the following statement.

Let C be an arbitrary Jordan curve, and let the function $u(x, y)$ be harmonic on and within C. Then there exists a sequence of harmonic polynomials $\{p_n(x, y)\}$ of respective degrees n coinciding with $u(x, y)$ at n points P_i of C, which converge uniformly on and within C to the function $u(x, y)$. These n points P_i can be chosen as points of C which correspond, under conformal mapping of the exterior of C on the exterior of a circle K so that the points at infinity correspond to the vertices of a regular polygon of n sides inscribed in K.

Let the function $u(x, y)$ be harmonic in the closed region C bounded by two analytic Jordan curves C_1 and C_2, with C_2 interior to C_1 and the origin interior to C_2. Green's formula

$$u(x,y) = \frac{1}{2\pi}\int_{C_1+C_2}\left[u\frac{\partial \log r}{\partial n} - \log r\frac{\partial u}{\partial n}\right]ds,$$

where the integrals are to be taken in the positive sense with respect to the region, n being the inner normal, breaks up the function $u(x, y)$ into a function harmonic on and interior to C_1, plus a function harmonic on and exterior to C_2, plus a multiple of $\log r$, where $r^2 = x^2 + y^2$ [see Osgood, 1, pp. 642–644, Walsh, 1, p. 206]. Consequently the function $u(x, y)$ can be approximated in the closed region C as closely as desired by a harmonic polynomial in (x, y) plus a harmonic polynomial in

$$\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

plus a multiple of $\log r$. A similar situation obtains if C is bounded not by two Jordan curves but by n Jordan curves, and is not materially altered if these bounding curves are not analytic, nor if $u(x, y)$ is not harmonic in the closed region, but is continuous in the closed region and harmonic in the interior.

We are in a position however to establish a more general* theorem.

Let C be a closed point set which does not contain the point at infinity and which contains no region of infinite connectivity not included in a larger region of finite connectivity belonging to C. Let $f(x, y)$ be an arbitrary function continuous on C and

* More general, that is, so far as concerns the point sets and functions considered; actually less specific in one regard, for the older theorem splits up the given function into $2n-1$ functions, of which $n-1$ are logarithms and left unchanged in the approximation. The other n functions are harmonic in simply-connected regions in whose interiors C lies, and *each* of these n functions can be uniformly approximated or expanded in terms of harmonic rational functions in the corresponding simply-connected region.

harmonic in the interior points of C. Then if an arbitrary positive ϵ be given, there exists a harmonic function $\phi(x, y)$, namely, a polynomial in (x, y) and in

$$\left(\frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2}, \frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2}\right),$$

such that we have

$$(11) \quad \left| f(x,y) - \phi(x,y) - a_1 \log r_1 - a_2 \log r_2 - \cdots - a_m \log r_m \right| < \epsilon, \quad (x,y) \text{ in } C,$$

where the a_i are suitable constants and $r_i^2 = (x-x_i)^2 + (y-y_i)^2$. Here the points (x_i, y_i), $i = 1, 2, \cdots$, are exterior to C, and can be preassigned, one in each of the regions R into which C separates the plane, although all of such preassigned points do not necessarily appear in (11).

In particular if C has no interior points, an arbitrary function $f(x, y)$ continuous on C can be so approximated.

Let us outline the proof; the method is essentially due to Lebesgue. Let K be a circle which contains C. There exists a function $F(x, y)$ continuous on and within K and which coincides with $f(x, y)$ on C. There exists a polynomial $p(x, y)$ in (x, y) which is not necessarily harmonic but which throughout K differs from $F(x, y)$ by less than $\epsilon/3$. Divide the plane into squares and continue subdivision indefinitely, by halving the sides of the squares already constructed, so that we determine a sequence of closed point sets S_i, each of which consists of a finite number of regions each bounded by a finite number of non-intersecting Jordan curves, of such a nature that each S_i contains C in its interior, so that each S_i contains its successors, but so that every point exterior to C is exterior to some S_i. By suitable modification of these point sets S_i if necessary, we can make sure that no two of the closed regions composing any particular S_i have a point in common. Let $h_i(x, y)$ be the function harmonic throughout the interior of S_i, continuous on S_i, and coinciding on the boundary of S_i with the function $p(x, y)$.

Then we have [Walsh, 1, p. 199] $\lim_{i=\infty} h_i(x, y) = p(x, y)$ uniformly on the boundary points of C. That is, in particular we can choose k so that $|h_k(x, y) - p(x, y)| < \epsilon/3$ uniformly on the boundary of C. But on C the function $h_k(x, y)$ can be uniformly approximated by a function of the sort considered in the theorem:

$$(12) \quad | h_k(x, y) - \phi(x, y) - a_1 \log r_1 - \cdots - a_m \log r_m | < \frac{\epsilon}{3}, \ (x, y) \text{ on } C,$$

and the points (x_i, y_i) lie exterior to C. In fact, if the points (x_i, y_i) are not preassigned, we can [loc. cit., p. 208] satisfy (12) on the entire point set S_k. If the points (x_i, y_i) are preassigned, we can make the approximation (12) not on the point set S_k but on a point set S_k' of the same connectivity as S_k, bounded by a finite number of non-intersecting Jordan curves, which is contained in S_k, which contains C, but which contains precisely those of the preassigned points (x_i, y_i) which lie in regions R lying entirely in S_k. In this new approximation (12), which holds in S_k' and hence in C, we use only the preassigned points (x_i, y_i).

Combination of the inequalities obtained yields (11) uniformly for all points on the boundary of C. But all the functions in the left-hand member of (11) are harmonic in the interior points of C, continuous on the corresponding closed point set, and such a function has no maximum or minimum in an interior point. Hence inequality (11) holds for *all* points of C, and the theorem is established.

We mention explicitly that it is not true that an arbitrary function continuous in a limited closed region and harmonic interior to the region can be uniformly approximated in that region as closely as desired by a harmonic rational function, either with or without logarithmic terms as considered in (11). Consider for example the region C formed from the circle $x^2+y^2<1$ by cutting out the line segment $y=0$, $-1/2 \leqq x \leqq 1/2$; let $f(x, y)$ be the function harmonic interior

to this region, continuous in the closed region, zero on the circumference, and unity on the line segment. The function $f(x, y)$ cannot be uniformly approximated in the closed region. For the approximating functions have only isolated singularities [Osgood 1, p. 680], are continuous without exception in the closed region (else are not uniformly bounded), hence are harmonic in the closed region. These approximating functions can be uniformly approximated as closely as desired in the closed region C by harmonic polynomials in (x, y), so $f(x, y)$ can also be uniformly approximated in C as closely as desired by a harmonic polynomial in (x, y). But if such a polynomial differs from $f(x, y)$ by less than ϵ for points on $x^2+y^2=1$, that polynomial differs from zero by less than ϵ for all points $x^2+y^2<1$, which is a contradiction for $x=y=0$ if $\epsilon<1/2$.

The general theorem we have proved, culminating in inequality (11), is obviously not an exhaustive discussion of its subject-matter. If C is a closed region of infinite connectivity, is it true that an arbitrary function $f(x, y)$ harmonic interior to C and continuous in the closed region, can in the closed region be uniformly approximated as closely as desired, as in (11)? Is our general theorem true without any restriction as to regions of infinite connectivity? Is it true that if C is an arbitrary closed point set without interior points, then an arbitrary function $f(x, y)$ continuous on C can be uniformly approximated on C as closely as desired by a rational harmonic function *without logarithmic terms*?

Even though we are not in a position to answer this last question, there are specific closed point sets C on which an arbitrary continuous function can certainly be approximated as closely as desired by a harmonic rational function. In fact, *an arbitrary closed point set C which consists of a finite number of Jordan arcs which do not divide the plane into an infinite number of regions has this property.* More generally, [compare Walsh, 5] *if C is such a point set that an arbitrary function continuous on C can be approximated on C as closely as desired by a rational function of the complex variable, then*

C also has the property considered. For let $f(x, y)$ be the given (real) function; a rational function $p(z)$ of z exists so that we have $|f(x, y)-p(z)|<\epsilon$, (x, y) on C, where ϵ is preassigned. Hence we have also $|f(x, y)-r(x, y)|<\epsilon$, (x, y) on C, where $r(x, y)$ is a rational harmonic function of (x, y), the real part of $f(z)$.

We do not attempt to approximate the most general function harmonic in a multiply-connected region C by harmonic rational functions without the use of logarithmic terms, for that is impossible, as we shall now prove, if the closed region cannot be considered as a closed simply-connected region.* We choose C limited, so there exists an analytic Jordan curve J interior to C, whose interior contains points not belonging to the closed region C; let us assume the origin to be such a point, so that the function $\log r$, where $r^2=x^2+y^2$, is harmonic in the closed region C. Assume the approximation possible, so that $\log r$ can be expanded in C in a uniformly convergent series of harmonic rational functions. These rational functions have only isolated singularities and are continuous in C, hence harmonic throughout the interior of C. Differentiate this series term by term in the direction of the normal to J, and integrate the resulting series over J term by term [Osgood 1, pp. 652–653]. For each term of the series, the result of this process is zero [Osgood 1, p. 680], but for the function $\log r$ the result is 2π. We thus have the contradiction $2\pi=0$, and the statement is established.

The entire discussion we have given enables us to answer the question as to whether a function given on an arbitrary closed point set C can be uniformly approximated on that point set by harmonic rational functions plus logarithmic terms, with singularities in assigned points not belonging to C, provided that no region of infinite connectivity is involved. The facts and proofs are so similar to the corresponding facts and proofs for the case of approximation by harmonic polynomials [Walsh, 1] that they are omitted.

* Already proved [Walsh 1, p. 206] in a special case.

It is not our purpose to study non-uniform expansion of harmonic functions, but one result is now interesting and yet so obvious that we mention it here; it is still an open problem to determine in general what functions can be expanded not necessarily uniformly in terms of harmonic rational functions either with or without logarithmic terms.

Let C_0 be an arbitrary open point set, which may be empty. Let $C_1, C_2, \cdots$ be closed point sets (any or all of which may be empty) mutually exclusive and having no common point with C_0, and on each of the point sets $C_0, C_1, \cdots$ let the function $u(x, y)$ be expansible by harmonic rational functions plus logarithmic terms. Then the function $u(x, y)$ can be expanded on $C_0+C_1+C_2+\cdots$ in a series of rational harmonic functions plus logarithmic terms. The series converges uniformly on each of the point sets $C_1, C_2, \cdots$ on which $u(x, y)$ is uniformly expansible, and if $u(x, y)$ is harmonic on C_0, except possibly for logarithmic singularities or singularities corresponding to rational harmonic functions, then the series converges uniformly on every closed point set contained in C_0.

We use $\psi(x, y)$ generically to denote a rational harmonic function with logarithmic terms, such as occurs in (11). Let $S_1, S_2, \cdots$ be closed point sets, each consisting of a finite number of mutually exclusive regions, each bounded by a finite number of non-intersecting Jordan curves, and such that S_k lies interior to C_0, that S_k lies interior to S_{k+1}, and that every point of C_0 lies in some S_k. Let us suppose

$$u(x,y) = \lim_{n\to\infty} \psi_{kn}(x,y), \qquad [(x,y) \text{ on } C_k,\ k = 0,1,2, \cdots],$$

where this limit is approached uniformly if $u(x, y)$ is uniformly expansible on C_k, $k \geqq 1$, and uniformly on every closed point set in C_0 for $k=0$ if that is possible. Then we can determine a closed point set B_{11}, consisting of a finite number of mutually exclusive closed regions each bounded by a finite number of non-intersecting Jordan curves, which contains C_1 but has no point in common with S_1. We can also determine $\psi_1(x, y)$ (see the lemma below) such that

$$|\psi_1(x,y) - \psi_{02}(x,y)| < \tfrac{1}{2}, \qquad (x,y) \text{ on } S_1,$$
$$|\psi_1(x,y) - \psi_{11}(x,y)| < \tfrac{1}{2}, \qquad (x,y) \text{ on } B_{11}.$$

Let B_{12} be a closed point set of the topological simplicity of B_{11}, which contains C_1 but has no point in common with S_2 or C_2. Let B_{21} be a closed point set likewise of the topological simplicity of B_{11}, which contains C_2 but has no point in common with S_2 or B_{12}. Choose $\psi_2(x, y)$ so that we have

$$|\psi_2(x,y) - \psi_{03}(x,y)| < \tfrac{1}{3}, \qquad (x,y) \text{ on } S_2,$$
$$|\psi_2(x,y) - \psi_{12}(x,y)| < \tfrac{1}{3}, \qquad (x,y) \text{ on } B_{12},$$
$$|\psi_2(x,y) - \psi_{21}(x,y)| < \tfrac{1}{3}, \qquad (x,y) \text{ on } B_{21}.$$

We continue this process; in general B_{1n} shall contain C_1 but shall have no point in common with S_n or C_2, C_3, $\cdots$, C_n. The point set $B_{2,n-1}$ shall contain C_2 but shall have no point in common with C_3, $\cdots$, C_n, S_n or B_{1n}. The point set B_{n1} shall contain C_n but shall have no point in common with S_n or B_{1n}, $B_{2,n-1}$, $\cdots$, $B_{n-1,2}$. The function $\psi_n(x, y)$ is then to be chosen so that we have

$$|\psi_n(x,y) - \psi_{0,n+1}(x,y)| < \frac{1}{n+1}, \quad (x,y) \text{ on } S_n,$$
$$|\psi_n(x,y) - \psi_{1n}(x,y)| < \frac{1}{n+1}, \quad (x,y) \text{ on } B_{1n},$$
. . . .
$$|\psi_n(x,y) - \psi_{n1}(x,y)| < \frac{1}{n+1}, \quad (x,y) \text{ on } B_{n1},$$

and the sequence $\{\psi_n(x, y)\}$ has the property required in the theorem. Uniform convergence of the sequence $\{\psi_n(x, y)\}$ on a closed point set belonging to $C_0+C_1+\cdots$ but not necessarily a C_k depends merely on the uniform convergence of the corresponding sequence (or sequences) $\{\psi_{kn}(x, y)\}$ on that point set. The function $u(x, y)$, if harmonic on C_0 except possibly for logarithmic singularities or singularities corresponding to rational harmonic functions, can be chosen arbitrarily on C_0, and the sequence $\{\psi_n(x, y)\}$ converges to the

value $u(x, y)$, uniformly on any closed point set contained in C_0.

If none of the point sets $C_0, C_1, C_2, \cdots$ separates any point of another of those point sets from the point at infinity the theorem is true if the words *harmonic rational functions plus logarithmic terms* are replaced by *harmonic polynomials*. For uniform convergence on any closed point set contained in C_0, however, we require that $u(x, y)$ should be harmonic in C_0 and that C_0 should be composed of mutually exclusive simply-connected regions. In the discussion just given we have had occasion to apply the following lemma.

LEMMA. *If S_1 and S_2 are mutually exclusive closed point sets each consisting of a finite number of mutually exclusive regions each bounded by a finite number of non-intersecting Jordan curves, and if the functions $\psi_1(x, y)$ and $\psi_2(x, y)$ are rational harmonic functions with logarithmic terms, then if $\epsilon>0$ be given there exists a rational harmonic function $\psi(x, y)$ with logarithmic terms so that we have*

$$|\psi(x,y) - \psi_1(x,y)| < \epsilon, \qquad (x,y) \text{ in } S_1,$$
$$|\psi(x,y) - \psi_2(x,y)| < \epsilon, \qquad (x,y) \text{ in } S_2.$$

For simplicity in the proof we assume that both S_1 and S_2 are limited. We write $\psi_1(x, y) = \psi_1'(x, y) + \psi_1''(x, y)$, where $\psi_1'(x, y)$ has no singularities in S_1 and $\psi_1''(x, y)$ has none in S_2. Similarly set $\psi_2(x, y) = \psi_2'(x, y) + \psi_2''(x, y)$, where $\psi_2'(x, y)$ has no singularities in S_2, and $\psi_2''(x, y)$ none in S_1; this splitting up is possible in the present case, but a slight modification may be necessary, due to the presence of logarithmic terms, if the point at infinity belongs to S_1 or S_2. It is possible [Walsh, 1, p. 208] to determine $\psi'(x, y)$ so that the inequalities

$$|\psi'(x,y) - \psi_1'(x,y) + \psi_2''(x,y)| < \epsilon, \qquad (x,y) \text{ in } S_1,$$
$$|\psi'(x,y) - \psi_2'(x,y) + \psi_1''(x,y)| < \epsilon, \qquad (x,y) \text{ in } S_2,$$

are satisfied. These inequalities, if we set

$$\psi(x,y) = \psi'(x,y) + \psi_1''(x,y) + \psi_2''(x,y),$$

are the inequalities it is desired to establish.

We mention explicitly the general question of the possibility of the expansion of given functions in terms of harmonic polynomials or in terms of harmonic rational functions when the restriction of uniformity of convergence is not made. This question seems not to be completely answered in the literature, although Lavrentieff [1] has recently announced some results without proof, and Hartogs and Rosenthal [1] have published an important paper on the corresponding subject for expansion in terms of polynomials in the complex variable. It is worth noting that Osgood's classical theorem [2] in this corresponding subject has the following analog in the present one.

If there converges in a region R a sequence of functions $\{u_n(x, y)\}$ *harmonic in R, then this sequence converges uniformly in some sub-region of R.*

From this theorem follows directly Osgood's better known result, that *if there converges in a region R a sequence of functions* $\{f_n(z)\}$ *analytic in that region, then this sequence converges uniformly in some sub-region of R.* For since the sequence $\{f_n(z)\}$ converges, the sequence $\{u_n(x, y)\}$ of the real parts of these functions converges in R, hence uniformly in some sub-region R' of R. Convergence of the sequence $\{iv_n(x, y)\}$ of the pure imaginary parts of the functions $f_n(z)$ at a single point of R' is now sufficient to ensure convergence of the sequence $\{iv_n\}$ and hence of the sequence $\{f_n(z)\}$ uniformly in any simply-connected closed proper sub-region of R'.

Let us indicate briefly the proof of the theorem for harmonic functions. We prove first, after Montel [1, p. 109], but this is only a modification of Osgood's proof, that the sequence $\{u_n(x, y)\}$ is uniformly bounded in some sub-region of R. Otherwise we should have some $|u_{n_1}(x, y)|>1$ at some point of R, hence in some sub-region R_1 of R. If the sequence $\{u_n(x, y)\}$ is not uniformly bounded in R_1, we must have $|u_{n_2}(x, y)|>2$ at some point of R_1, and therefore in some sub-region R_2 of R_1. Proceeding in this way we arrive at

a sequence $\{u_{n_k}(x, y)\}$ greater in absolute value than $\{k\}$ in R_k. There is at least one point common to all the R_k, and at this point the original sequence cannot converge.

From the boundedness of the original sequence in some sub-region R' it follows [compare Osgood, 2] that the first partial derivatives of these functions are uniformly bounded in an arbitrary closed sub-region of R', hence that the functions $\{u_n(x, y)\}$ are equicontinuous in this sub-region and therefore converge uniformly there.

We shall not go into great detail on the subject of rational harmonic functions *belonging to a region*, for the discussion can be made to depend upon the discussion for harmonic polynomials belonging to a region. Let a region C be bounded by the Jordan curves $C_0, C_1, \cdots, C_k$, of which no two have a common point, and so that the curves $C_1, C_2, \cdots, C_k$ lie interior to C_0. Let $u(x, y)$ be an arbitrary function harmonic interior to C and continuous in the closed region. As has already been suggested, Green's formula applied to $u(x, y)$ not for the region C but for a neighboring region C' of the same connectivity interior to C splits up the function $u(x, y)$ into k logarithmic terms with singularities at infinity and at points interior to $C_1, C_2, \cdots, C_k$ respectively, which may be preassigned, plus k functions harmonic exterior respectively to $C_1, C_2, \cdots, C_k$, including the point at infinity, and continuous in the corresponding closed regions, plus a function harmonic interior to C_0 and continuous in the corresponding closed region. If the curves $C_0, C_1, \cdots, C_k$ are suitably restricted, the regions interior to C_0 and exterior to $C_i (i = 1, 2, \cdots, k)$ respectively have associated with them (compare §3) sets of rational functions, in fact polynomials in (x, y) and in

$$\left(\frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2}, \frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2}\right)$$

respectively, and the last mentioned $k+1$ functions can be respectively expanded in terms of these functions.

It would be an interesting problem, however, to construct

for the region C *a single set* of normal orthogonal harmonic rational functions (with also k logarithmic functions) in terms of which an arbitrary harmonic function could be expanded. Here orthogonal may mean (1) with respect to area (that is, surface integral over C) or (2) with respect to length (that is, line integral over $C_0, C_1, \cdots, C_k$). Such a set of functions can easily be constructed by the process of orthogonalization, starting with the logarithmic and rational functions we have been using. If interpretation (2) is used, it is true that *if the curves* $C_0, C_1, \cdots, C_k$ *bounding the region* C *are analytic, and if the function* $u(x, y)$ *is harmonic interior to* C, *continuous in the corresponding closed region, then the formal expansion of* $u(x, y)$ *in terms of these normal orthogonal functions belonging to* C *converges to the value* $u(x, y)$ *throughout the interior of* C, *uniformly on any closed point set interior to* C. This follows from the reasoning as given by Merriman [1]; but questions of asymptotic character of the normal orthogonal functions and of uniform convergence of the formal expansion in the closed region C, are there left unanswered under the present general hypothesis on the region C.

6. *Degree of Approximation by Rational Harmonic Functions.* Our results on the degree of approximation to a harmonic function by rational harmonic functions are to be obtained with the help of the corresponding results on the degree of approximation to an analytic function by rational functions of the complex variable, so as a preliminary study we need to consider the relation between the degree of a rational harmonic function and the degree of the corresponding analytic function. In this discussion we use the word *degree* to indicate degree *in the restricted sense.**

If $u(x, y)$ is a harmonic polynomal of degree n, then its conjugate function $v(x, y)$ is likewise a harmonic polynomial of degree n and hence the function $f(z) = u(x, y) + iv(x, y)$ is a

* Compare §4. The degree of a rational function is the greater of the degrees of numerator and denominator, or the common degree if the two are of the same degree.

polynomial in z of degree n. In fact, we may define $v(x, y)$ by

$$v(x,y) = \int_{(0,0)}^{(x,y)} \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \right) + C,$$

where C is an arbitrary constant. The function $f(z) = u(x, y) + iv(x, y)$ can have no singular points other than those of $u(x, y)$, hence is an entire function. In the Taylor's development of $f(z)$ about the origin, the coefficients of powers of z higher than the nth vanish; otherwise $u(x, y)$ is not a polynomial of degree n.

The situation is somewhat more complicated if $u(x, y)$ is a harmonic rational function not a polynomial. Osgood [1, p. 680] has shown that if $u(x, y)$ is harmonic and rational, then $f(z) = u(x, y) + iv(x, y)$, where $v(x, y)$ is defined by the equation just given, is a rational function of z. If $f(z)$ is given, with no factor containing z common to numerator and denominator,

$$f(z) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}, \qquad a_0 b_0 \neq 0,$$

the real part is found by multiplying numerator and denominator by the conjugate complex quantity of the original denominator,

$$f(z) = \frac{a_0 \bar{b}_0 z^m \bar{z}^n + \cdots + a_m \bar{b}_n}{b_0 \bar{b}_0 z^n \bar{z}^n + \cdots + b_n \bar{b}_n},$$

and then separating into real and pure imaginary parts. The function $u(x, y)$ is then a rational function whose numerator is of total degree $m+n$ or less, whose denominator is of degree $2n$, and with no factor common to numerator and denominator; the actual degree of $u(x, y)$ is $m+n$ or less, or $2n$ according as $m \geqq n$ or $m < n$. Reciprocally, let $u(x, y)$ be given in its lowest terms; denote the degrees of numerator and denominator by p and $2q$ respectively, and those of the numerator and denominator of the corresponding analytic function $f(z)$, which is determined only to within an additive constant, by m and n. A necessary and sufficient condition

that $u(x, y)$ be harmonic at infinity is that $f(z)$ be analytic at infinity, that is, $m \leqq n$; in this case we have $n = q$, $m \geqq p - q$; the actual degree of $u(x, y)$ is $2n$ and that of $f(z)$ is n. A necessary and sufficient condition that $u(x, y)$ be singular at infinity is that $f(z)$ be singular at infinity, that is, $m > n$. Here we may write $f(z)$ as a rational function whose numerator is of degree $n-1$ or less and denominator of degree n plus a polynomial of degree $m-n$. Then $u(x, y)$ is a rational function whose numerator is of degree $2n-1$ or less and denominator of degree $2n$ plus a polynomial of degree $m-n$, or a rational fraction whose numerator is of degree $p = m+n$ and denominator of degree $2q = 2n$; the actual degree of $u(x, y)$ is $m+n$ and that of $f(z)$ is m.

We can prove the following result concerning approximation by harmonic rational functions; henceforth we use the word *degree* in its inclusive sense.

Let S be an arbitrary closed Jordan region of the (x, y)-plane. If there exist rational harmonic functions $r_n(x, y)$, real parts of rational functions of $z = x + iy$ of respective degrees n such that we have*

$$| u(x, y) - r_n(x, y) | \leqq \frac{M}{R^n}, \qquad R > 1,$$

for all points (x, y) of S and for all sufficiently large n, and if the singular points of the functions $r_n(x, y) - r_{n-1}(x, y)$ have no limit point on S, then $u(x, y)$ is the real part of a function $f(z)$ meromorphic on S. If in addition the functions $r_n(x, y)$ have no singularities on S, then $u(x, y)$ is harmonic on S.

Let $w = \Phi(z)$ denote a function which maps the complement of S onto the exterior of the unit circle in the w-plane, so that the points at infinity correspond to each other. Let S_R denote the Jordan curve $|\Phi(z)| = R$, where $R > 1$. If the singularities of the functions $r_n(x, y) - r_{n-1}(x, y)$ have no limit point interior to

* The rational function of z is not determined uniquely by $r_n(x, y)$, but merely to within an additive constant. Nevertheless the degree, and nature and location of the singularities of this rational function are determined by $r_n(x, y)$ uniquely.

S_ρ, then the sequence $\{r_n(x, y)\}$ converges interior to S_ν, where $\nu=(1+\rho R^{1/2})/(\rho+R^{1/2})$, and the convergence is uniform on any closed point set interior to S_ν. Hence $f(z)$ is meromorphic interior to S_ν and if $r_n(x, y)$ has no singularities interior to S_ν, the function $u(x, y)$ is harmonic interior to S_ν.

If the function $r_n(x, y)-r_{n-1}(x, y)$ is the real part of a function which has at most n poles, for n sufficiently large, we may set $\nu=(1+\rho R)/(\rho+R)$, and in particular if the only singularities of $r_n(x, y)$ lie at infinity, we may set $\rho=\infty$, $\nu=R$, so that the sequence $\{r_n(x, y)\}$ converges interior to S_R.

The proof follows the proof of the corresponding theorem [Walsh 3, Theorem I] for approximation by means of polynomials. If the functions $f(z)$ and $t_n(z)$, of which $u(x, y)$ and $r_n(x, y)$ are respectively the real parts, are properly chosen, and if S' is an arbitrary closed region interior to S, then the inequality $|f(z)-t_n(z)|\leqq M'/R^n$ is valid for z in S'. If the region S' is a region bounded by a Jordan curve uniformly near the boundary of S, the function $w=\Phi'(z)$ which maps the complement of S' onto the exterior of the unit circle in the w-plane differs little from the function $w=\Phi(z)$, and the curve $S_R': |\Phi'(z)|=R$ lies uniformly near the curve $S_R: |\Phi(z)|=R$. The theorem follows by virtue of the corresponding theorem for approximation to analytic functions by means of rational functions [Walsh 4, Theorem IV].

It is a positive simplification here not to mention the degree of $r_n(x, y)$ except in connection with $t_n(z)$, for such widely differing functions as

$$x, \quad \frac{x}{x^2+y^2}, \quad \frac{3x^2+3y^2+x}{x^2+y^2}$$

belong in the same category.

We turn now to the study of the approximation of functions by rational harmonic functions in multiply connected regions. Here logarithmic terms are essential for the approximation of the most general harmonic functions, as we have seen, so we prove the following theorem.

Let C be a closed region bounded by Jordan curves C_0, C_1, $\cdots$, C_k, such that no two of these curves have a common point and so that C_1, C_2, $\cdots$, C_k lie interior to C_0. If the sequence

$$\begin{aligned} &r_n(x,y) + A_{n1}\log\left[(x-x_{n1})^2+(y-y_{n1})^2\right] \\ &+ A_{n2}\log\left[(x-x_{n2})^2+(y-y_{n2})^2\right] \\ &+ \cdots + A_{nk}\log\left[(x-x_{nk})^2+(y-y_{nk})^2\right], \\ &\qquad\qquad (n=1,2,\cdots), \end{aligned} \tag{13}$$

converges uniformly in C, where $r_n(x,y)$ is a harmonic rational function of (x,y) with no singularities interior to C, and where the point (x_{ni}, y_{ni}) lies in or on C_i and approaches a limit (x_i, y_i) as n becomes infinite, then $\lim_{n\to\infty} A_{ni}$ exists, $i=1,2,\cdots, k$, and each of the sequences*

$$r_n(x,y),\ A_{ni}\log\left[(x-x_{ni})^2+(y-y_{ni})^2\right],\ (i=1,2,\cdots,k), \tag{14}$$

converges uniformly in any closed region interior to C; if the functions in (14) *have no singularity in C and no limit point of singularities in C, this convergence is uniform in the closed region C.*

The proof is simple; choose an analytic Jordan curve C_i' interior to C and enclosing C_i but none of the curves C_1, C_2, $\cdots$, C_{i-1}, C_{i+1}, $\cdots$, C_k in its interior. The sequence of the derivatives of (13) in the direction of the normal ν to C_i' converges uniformly on C_i', and can be integrated on C_i' term by term [Osgood 1, pp. 652–653, Walsh 1, p. 206]. It follows from the theorem of Osgood already quoted that

$$\int_{C_i'} \frac{\partial r_n(x,y)}{\partial \nu} ds = 0\,;$$

for we have

$$\int_{C_i'} \frac{\partial r_n(x,y)}{\partial \nu} ds = \int_{C_i'} \frac{\partial s_n(x,y)}{\partial s} ds = 0,$$

* It is of course sufficient if the point (x_i, y_i) lies on or within C_i without the assumption that (x_{ni}, y_{ni}) lies interior to or on C_i; we integrate (as below) over C_i' when n is chosen so large that all points (x_{ni}, y_{ni}) but no points (x_{nj}, y_{nj}), $i \neq j$, lie interior to C_i'.

where $s_n(x, y)$ is a function conjugate to $r_n(x, y)$. The total result of the integration of the nth term of the sequence (13) is then precisely $4\pi A_{ni}$, so $\lim_{n\to\infty} A_{ni} = A_i$ exists. It follows that

$$\lim_{n\to\infty} A_{ni} \log [(x - x_{ni})^2 + (y - y_{ni})^2] = A_i \log [(x - x_i)^2 + (y - y_i)^2],$$

uniformly in any closed region interior to C, and hence it follows that the sequence $\{r_n(x, y)\}$ converges uniformly as asserted.

In the next theorem we shall not trouble to consider sequences so general as (13).

Let C be a closed region bounded by Jordan curves C_0, C_1, $\cdots$, C_k, such that no two of these curves have a common point and such that $C_1, C_2, \cdots, C_k$ lie interior to C_0. A necessary and sufficient condition that the function $u(x, y)$ defined in C should be harmonic in (the closed region) C, is that there should exist rational harmonic functions $r_n(x, y)$, real parts of rational functions of z of degrees $(k+1)n$, $n = 0, 1, 2, \cdots$, so that we have

$$(15) \quad \left| u(x, y) - \sum_{i=1}^{k} A_{in} \log [(x - x_i)^2 + (y - y_i)^2] - r_n(x, y) \right| \leqq \frac{M}{R^n}, \quad R > 1,$$

for all points (x, y) of C; here the A_{in} are constants and the points (x_i, y_i) are supposed to lie interior to C_i respectively. It follows that $\lim_{n\to\infty} A_{in} = A_i$ exists if (15) is satisfied.

The sufficiency of this condition follows, as in the proof already given, from Osgood's theorem and from the corresponding result for approximation of analytic functions of the complex variable [Walsh 4, Theorem V]. It is found by the integration that

$$|A_{in} - A_i| \leqq M'/R_n;$$

hence (15) holds if A_{in} is replaced by A_i, provided that M is replaced by a suitable M''. The necessity of the condition is likewise easy to establish. The points (x_i, y_i) may be chosen arbitrarily interior to the curves C_i respectively. Then [com-

pare Osgood 1, pp. 642–644; Walsh 1, p. 206] in C the function $u(x, y)$ may be expressed as the sum of the k logarithmic terms which appear in (15) plus k functions harmonic respectively on and exterior to C_i, and at infinity, for $i=1, 2, \cdots, k$, plus a function harmonic in the closed interior to C_0. The results of §4 can now be applied, and yield the desired rational functions $r_n(x, y)$.

This theorem can be extended by considering (1) variable logarithmic terms, (2) more explicit regions or point sets for the location of the singularities of the functions $r_n(x, y)$, (3) more explicit regions (that is, regions containing C in their interiors) for the harmonic character of $u(x, y)$ and for the convergence of the sequence $\{r_n(x, y)\}$, (4) more general boundaries for regions than Jordan curves. There is comparatively little difficulty involved in making any of these generalizations however, and the essential reasoning involved already appears in the literature (either here or elsewhere), so these generalizations are left to the reader.

It is also possible to study sequences of harmonic rational functions which satisfy such a relation as (15) not in a region but on a Jordan arc or curve. Here the situation is not so simple, as we have indicated in the analogous situation in §4. If (15) holds merely on the unit circle C, for instance, the sequence of harmonic rational functions need not converge elsewhere; for the function $A_n \log r$ vanishes on C and a change in A_n does not alter (15), yet may alter the convergence off of C of the sequence in (15). The rational harmonic functions are not uniquely determined, moreover, by their values on C; the two functions $r^n \cos n\theta$ and $r^{-n} \cos n\theta$ are equal on C. Knowledge even of the *location* of the singularities of a harmonic rational function (such as $ar^n \cos n\theta + br^{-n} \cos n\theta$) and of the value on C does not determine the rational function uniquely. Nor is a function $u(x, y)$ harmonic in a region containing C in its interior determined by its values on C.

If (15) holds on the unit circle C for rational functions $\{r_n(x, y)\}$ with singularities only at the origin and at infinity

and where (x_i, y_i), $i=1$, is the origin, there exists a function $U(x, y)$ harmonic on C, in fact harmonic for $r<R$, and coinciding on C with the values $u(x, y)$. Indeed, the logarithmic term of (15) vanishes on C and hence may be entirely omitted. The rational function $r_n(x, y)$ is a polynomial in

$$r^{-n}\cos n\theta, r^{-n}\sin n\theta, \cdots, 1, \cos\theta, \sin\theta, \cdots, r^n\cos n\theta, r^n\sin n\theta,$$

but the negative powers of r may be changed into positive powers without altering $r_n(x, y)$ or (15) on C, so we are dealing with a sequence of harmonic polynomials, converging like a geometric series on C. This situation has already been treated in §4, and the existence of the required function $U(x, y)$ is established there.

7. *Expansions in Three Dimensions.* The theory of the expansion of harmonic functions in three dimensions is not nearly so far developed as the corresponding theory in two dimensions, but we shall discuss a few results and a few open problems.

It is classic that Poisson's integral yields directly an expansion of functions which are given harmonic interior to the sphere, continuous in the corresponding closed region.

If the function $u(x, y, z)$ is harmonic interior to a sphere and continuous in the corresponding closed region, then interior to the sphere $u(x, y, z)$ can be expanded in a series of harmonic polynomials in (x, y, z), the series converging uniformly in any closed region interior to the sphere.

This theorem will be used in proving the analog of the corollary (§2) to Runge's theorem. This analog has already been proved by Bergmann [1], but only for the case of convex regions. The method we shall use is closely related to that of Bergmann, and both methods are intimately related to that of Runge.

If the function $u(x, y, z)$ is harmonic in a closed region S bounded by a simple closed (limited) surface, then in that closed region the function $u(x, y, z)$ can be approximated uni-

formly as closely as desired by a harmonic polynomial in (x, y, z).

We shall find it convenient to have for reference the following lemma.

LEMMA. *Let the two points* (x_0, y_0, z_0) *and* (x', y', z') *and the simple polygonal curve C joining them lie exterior to the closed point set S. If the function* $u(x, y, z)$ *can be uniformly approximated in S as closely as desired by a harmonic function whose only singularity lies in* (x_0, y_0, z_0), *then this function* $u(x, y, z)$ *can likewise be uniformly approximated in S as closely as desired by a harmonic function whose only singularity lies in* (x', y', z') *and which is rational except for the factor* $[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}$.

We give the proof of this lemma for the case that both (x_0, y_0, z_0) and (x', y', z') are finite points, but that is simply a matter of convenience. The proof holds with only obvious changes even if (x', y', z') is the point at infinity,* and it is in this latter form that the lemma will be applied later. When (x', y', z') is the point at infinity, the approximating functions are polynomials in (x, y, z), and the factor $[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}$ does not enter.

Construct a finite sequence of spheres $S_0, S_1, \cdots, S_n$ extending from (x_0, y_0, z_0) to (x', y', z'), so that no sphere contains in its interior a point of S, but so that successive spheres of the sequence have a region common to them. The point (x_0, y_0, z_0) shall lie interior to S_0 and so shall another particular point (x_1, y_1, z_1) of C. The sphere S_1 shall contain (x_1, y_1, z_1) in its interior and likewise another particular point (x_2, y_2, z_2) which lies on C between (x_1, y_1, z_1) and (x', y', z'). We proceed in this way to construct $S_2, S_3, \cdots, S_{n-1}$, finally arriving at the sphere S_n which contains in its interior (x_n, y_n, z_n) and (x', y', z'). The construction of these spheres is surely possible, for it is no loss of generality to assume that C is composed of merely a finite number of line

* It is to be remembered that vanishing at infinity is a necessary condition that a function should be harmonic there.

segments and has the length l. Assume likewise that the nearest distance from C to a point of S is δ. Choose the sequence of points (x_0, y_0, z_0), (x_1, y_1, z_1), $\cdots$, (x_n, y_n, z_n), (x', y', z') on C such that the greatest distance between two successive points is less than $\delta/2$. The sphere S_i may be taken as the sphere with center (x_i, y_i, z_i) and diameter δ.

Let an arbitrary positive ϵ be given. We choose the harmonic function $r_0(x, y, z)$, whose only singularity lies in (x_0, y_0, z_0), such that we have

$$\left| u(x,y,z) - r_0(x,y,z) \right| < \frac{\epsilon}{n+2}, \qquad (x,y,z) \text{ in } S.$$

This choice is possible by hypothesis. We next choose the harmonic function $r_1(x, y, z)$, whose only singularity lies in (x_1, y_1, z_1), rational except for the factor $[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2]^{-1/2}$, such that we have

$$\left| r_0(x,y,z) - r_1(x,y,z) \right| < \frac{\epsilon}{n+2}, \qquad (x,y,z) \text{ in } S. \tag{16}$$

We prove by means of an inversion in the unit sphere whose center is (x_1, y_1, z_1) that such choice is possible:

$$\xi = \frac{x - x_1}{r^2}, \quad \eta = \frac{y - y_1}{r^2}, \quad \zeta = \frac{z - z_1}{r^2}, \tag{17}$$

$$r^2 = \frac{1}{\rho^2} = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

The function $\rho_0(\xi, \eta, \zeta) = (1/\rho)\, r_0(x, y, z)$, where $\xi, \eta, \zeta, x, y, z$ are connected by the relations (17), is a harmonic function of (ξ, η, ζ) on and within S_0', the transform of the sphere S_0 under the inversion. Then by the theorem already stated, the function $r'(\xi, \eta, \zeta)$, a polynomial in (ξ, η, ζ), can be determined so that we have

$$\left| \frac{1}{\rho} r_0(x,y,z) - r'(\xi,\eta,\zeta) \right| < \frac{\epsilon}{(n+2)d}, \ (\xi,\eta,\zeta) \text{ interior to } S_0',$$

where d is the distance from (x_1, y_1, z_1) to the farthest point of S_0'. This inequality implies the inequality

$$| r_0(x,y,z) - \rho r'(\xi,\eta,\zeta) | < \frac{\epsilon}{n+2}, \quad (x,y,z) \text{ exterior to } S_0.$$

But $(1/r)r'(\xi, \eta, \zeta)$ is a harmonic function of (x, y, z), rational except for the factor $1/r$ and with its only singularity in the point (x_1, y_1, z_1); hence its equal $\rho r'(\xi, \eta, \zeta)$ may be identified with the desired function $r_1(x, y, z)$ and yields (16).

This same process can now be continued, and yields the additional inequalities

$$| r_1(x,y,z) - r_2(x,y,z) | < \frac{\epsilon}{n+2}, \qquad (x,y,z) \text{ in } S,$$

. . . . . .

$$| r_n(x,y,z) - r_{n+1}(x,y,z) | < \frac{\epsilon}{n+2}, \qquad (x,y,z) \text{ in } S.$$

Hence we have finally

$$| u(x,y,z) - r_{n+1}(x,y,z) | < \epsilon, \qquad (x,y,z) \text{ in } S,$$

where $r_{n+1}(x, y, z)$ represents a harmonic function whose only singularity lies in (x', y', z') and is, except for the factor $[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}$, rational in (x, y, z). That is, the lemma is established.

We need a second lemma in the proof. For the sake of simplicity we state and prove the lemma for integrals of a function of a single variable, but the proof holds without essential change for multiple integrals, and to double integrals we shall apply the lemma. Likewise in the proof we consider merely a single parameter α, but no difficulty is introduced by the appearance of several, and several parameters appear in the application we shall make.

LEMMA. *Let $f(x, \alpha)$ be a continuous real function of the arguments for $a \leqq x \leqq b$, $\alpha_1 \leqq \alpha \leqq \alpha_2$. If an arbitrary positive δ be given, then there exists δ' so that the inequality*

$$(18) \qquad \left| \int_a^b f(x,\alpha)dx - \sum_{i=1}^{n} f(\xi_i,\alpha)(x_i - x_{i-1}) \right| \leqq \delta,$$

$$a = x_0 \leqq x_1 \leqq \cdots \leqq x_n = b, \qquad x_{i-1} \leqq \xi_i \leqq x_i,$$

holds uniformly for all α, $\alpha_1 \leqq \alpha \leqq \alpha_2$, *provided merely that we have* $|x_i - x_{i-1}| \leqq \delta'$.

By the mean value theorem for integrals, we can write

$$\int_a^b f(x,\alpha)dx = \sum_{i=1}^{n} f(\eta_i,\alpha)(x_i - x_{i-1}),$$

where $x_{i-1} \leqq \eta_i \leqq x_i$, and where η_i depends on α. Let us choose δ' so small that the inequality $|\zeta_i - \zeta_i'| \leqq \delta'$ implies

$$|f(\zeta_i,\alpha) - f(\zeta_i',\alpha)| \leqq \delta/(b-a)$$

uniformly for all α, $\alpha_1 \leqq \alpha \leqq \alpha_2$; such a δ' exists by the uniform continuity of $f(x, \alpha)$. Then under the hypothesis $|x_i - x_{i-1}| \leqq \delta'$, we have also $|\xi_i - \eta_i| \leqq \delta'$, from which follows the inequality

$$\left| \sum_{i=1}^{n} f(\eta_i,\alpha)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(\xi_i,\alpha)(x_i - x_{i-1}) \right| \leqq \delta,$$

which is precisely (18).

We are now in a position to prove the theorem. Denote by σ a simple closed surface consisting entirely of a finite number of portions of planes parallel to the coordinate planes, which contains S in its interior, and on and within which the given function $u(x, y, z)$ is harmonic. The value of $u(x, y, z)$ in S is given by Green's integral

$$(19) \qquad u(x,y,z) = \frac{1}{4\pi}\int_\sigma\int \left[\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial n} \right] d\sigma,$$

where n represents the exterior normal. Let an arbitrary positive ϵ be given. We divide σ into a finite number n of pieces σ_i, so that each piece consists only of a portion of a plane parallel to one of the coordinate planes, so that each piece σ_i lies interior to a sphere S_i which contains on or within it no point of S, and so that we have

$$\left|\frac{1}{4\pi}\int_\sigma\int \frac{1}{r}\frac{\partial u}{\partial n}d\sigma - \frac{1}{4\pi}\sum_{i=1}^{n}\frac{1}{r_i}\left(\frac{\partial u}{\partial n}\right)_i \Delta_i\sigma\right| \leqq \frac{\epsilon}{4},$$

$$(x,y,z) \text{ in } S,$$

(20)

$$\left|\frac{1}{4\pi}\int_\sigma\int u\frac{\partial(1/r)}{\partial n}d\sigma - \frac{1}{4\pi}\sum_{i=1}^{n}u_i\left[\frac{\partial(1/r)}{\partial n}\right]_i \Delta_i\sigma\right| \leqq \frac{\epsilon}{4},$$

$$(x,y,z) \text{ in } S,$$

where r_i is the distance from (x, y, z) to a particular but arbitrary point P_i of σ_i, where $(\partial u/\partial n)_i$ is the normal derivative of $u(x, y, z)$ at P_i, where $\Delta_i\sigma$ is the area of σ_i, where u_i is the value of $u(x, y, z)$ at P_i, and where $[\partial(1/r)/\partial n]_i$ is the value of $\partial(1/r)/\partial n$ at P_i. The last expression may also be written as the value at P_i of a partial derivative of $1/r$ with respect not to (x, y, z) but to running coordinates on σ, and this latter form serves better to indicate that we are dealing with a harmonic function of (x, y, z).

Each term of the sums in (20) is a function of (x, y, z) harmonic not merely in the closed region S but throughout the exterior of the corresponding sphere S_i, even at infinity. It is immediately seen by inversion with P_i as center of inversion that in the (x, y, z)-space each of these terms can be approximated by a harmonic function of (x, y, z) whose only singularity lies in P_i and which is rational except for the factor $1/r_i$, where r_i indicates distance measured from P_i. The approximation can be made uniformly as closely as desired in the exterior of the sphere S_i, and hence uniformly as closely as desired in the closed region S. But each point P_i can be joined to the point at infinity by a Jordan curve which does not meet S. Then by the first lemma we can approximate each term of the sums in (20) as closely as desired uniformly in the closed region S by a harmonic polynomial in (x, y, z). That is, we can determine harmonic polynomials $p_1(x, y, z)$ and $p_2(x, y, z)$ such that we have

$$\left|\frac{1}{4\pi}\sum_{i=1}^{n}\frac{1}{r_i}\left(\frac{\partial u}{\partial n}\right)_i \Delta_i\sigma - p_1(x,y,z)\right| \leqq \frac{\epsilon}{4}, \quad (x,y,z) \text{ in } S,$$

(21)

$$\left|\frac{1}{4\pi}\sum_{i=1}^{n}u_i\left(\frac{\partial(1/r)}{\partial n}\right)_i \Delta_i\sigma - p_2(x,y,z)\right| \leqq \frac{\epsilon}{4}, (x,y,z) \text{ in } S.$$

It follows immediately that we have

$$(22)\quad \left| u(x,y,z) - [p_1(x,y,z) - p_2(x,y,z)] \right| \leqq \epsilon,\ (x,y,z) \text{ in } S,$$

and the theorem is proved.

The present proof has the advantage over the proof of §2 of holding (with only minor modifications) in two, three, or n dimensions, and since it does not involve the theory of functions of a complex variable, has the additional advantage of being more satisfactory from the standpoint of the purist. This remark applies also to the results which immediately follow.

Another remark is of interest. If the point (x, y, z) is exterior to the closed surface σ, the integral (19) has the value zero, where r still indicates distance from (x, y, z) to an arbitrary point of σ. Indeed it is a general theorem that if the functions u and v are both harmonic on and within σ, then we have

$$\int_\sigma \int \left[v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right] d\sigma = 0.$$

Thus if the function $u(x, y, z)$ is given harmonic on a point set S composed of the closed interiors of two non-intersecting simple closed surfaces, two new non-intersecting simple closed surfaces σ_1 and σ_2 can be constructed which consist entirely of a finite number of portions of planes parallel to the coordinate planes, such that each point of S lies interior either to σ_1 or σ_2, and such that $u(x, y, z)$ is harmonic on and within both σ_1 and σ_2. Then formulas (19), (20), (21), (22) are valid without any change whatever, if we set $\sigma = \sigma_1 + \sigma_2$. A similar fact holds if S falls into n distinct parts. In fact we can prove the following theorem.

Let S be an arbitrary closed limited point set whose complementary set (with respect to the entire space) is connected. Then if $u(x, y, z)$ is harmonic on S, the function $u(x, y, z)$ can be uniformly approximated on S as closely as desired by a harmonic polynomial in (x, y, z).

There exists a point set σ consisting of a finite number of mutually exclusive closed limited simply-connected regions

$\sigma_1, \sigma_2, \cdots, \sigma_n$ containing all points of S in their interiors, the individual regions bounded entirely by a finite number of portions of planes parallel to the coordinate planes, and such that $u(x, y, z)$ is harmonic on the entire set $\sigma_1+\sigma_2+\cdots+\sigma_n$. The previous method applies without change and the theorem follows.

The following application corresponds to the theorem of Runge; the proof is omitted.

Let C be a point set composed of a finite or infinite number of mutually exclusive simply-connected regions, none of which contains the point at infinity in its interior. If the function $u(x, y, z)$ is harmonic interior to each of these regions, then $u(x, y, z)$ can be expanded on C in a series of harmonic polynomials in (x, y, z), and the series converges uniformly on any closed point set contained in C.

We add one further result on general approximation. A closed region S is said to be *convex with respect to the interior point P* if S is bounded by a surface which is cut in a single point by each ray through P. A region which is convex in the usual sense of the word is convex with respect to each of its interior points.

If the region S is a limited closed region convex with respect to some interior point P, then an arbitrary function $u(x, y, z)$, harmonic interior to S and continuous in the closed region, can be approximated as closely as desired uniformly in the closed region S by a harmonic polynomial in (x, y, z).

Choose P as origin of coordinates, which involves no loss of generality, and consider the transformation

$$x' = \rho x, \qquad y' = \rho y, \qquad z' = \rho z, \qquad \rho > 1,$$

which transforms S into a region S' which contains S in its interior. The function $u(x, y, z)$ is transformed into a new function

$$u'(x, y, z) = u\left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right)$$

defined throughout S'. It follows from the uniform contin-

uity of $u(x, y, z)$ in S that if $\epsilon > 0$ be given, then ρ can be chosen so near to unity that we have

$$| u'(x,y,z) - u(x,y,z) | < \frac{\epsilon}{2}, \qquad (x,y,z) \text{ in } S.$$

For $u'(x, y, z) - u(x, y, z)$ is the same as $u(x/\rho, y/\rho, z/\rho) - u(x, y, z)$, which becomes uniformly small with $\rho - 1$. The function $u'(x, y, z)$ is harmonic in the closed region S, so there exists a harmonic polynomial $p(x, y, z)$ such that we have

$$| u'(x,y,z) - p(x,y,z) | < \frac{\epsilon}{2}, \qquad (x,y,z) \text{ in } S,$$

whence follows

$$| u(x,y,z) - p(x,y,z) | < \epsilon, \qquad (x,y,z) \text{ in } S,$$

and the theorem is established.*

It will be noticed that the study of the approximation to an arbitrary function in the plane is fairly well developed, while for space the study is only begun. In particular the general question as to when a function harmonic interior to a three-dimensional region and continuous in the closed region (or more generally, defined on a given closed point set) can be uniformly approximated in that closed region (or on that point set) as closely as desired by a harmonic polynomial, seems to be well worth investigating. There is an essential difference in methods and results for two and for three dimensions, for in the latter case the Dirichlet problem does not always have a solution, even for simply-connected regions. Methods of solution of the problem of approximation in three dimensions would presumably depend on the study of the solution of the Dirichlet problem for variable regions, and the result would presumably depend to some extent on whether the region is normal [compare Kellogg, 1].

* The corresponding theorem for analytic functions of a complex variable was given independently by Hilb and Szász, Encyklopädie der mathematischen Wissenschaften, vol. 2, C, II, p. 1276, and by Walsh, Transactions of this Society, vol. 26 (1924), pp. 155–170; p. 168, footnote.

We mention other problems for three or more dimensions: the study of harmonic polynomials belonging to a general region*, the degree of approximation to functions by harmonic polynomials, the question of interpolation by harmonic polynomials, approximation by harmonic rational functions,—the solution of all of these problems still lies in the future.

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* For convex regions a treatment is given by Bergmann, 2.

IRREDUNDANT SETS OF POSTULATES FOR THE LOGIC OF PROPOSITIONS*

BY B. A. BERNSTEIN

1. *Introduction.* Alonzo Church† has introduced the interesting notion of "irredundance" in connection with a set of postulates. A set of postulates is irredundant "if the postulates are independent and no one of them can be weakened with respect to the set." I give in this paper a number of irredundant sets of postulates for the logic of propositions.‡

Church gives a "mechanical" method by which any set of independent postulates can be made irredundant: if

(I) $$A, B, C$$

is a set of independent postulates, the set

(II) *A, if A then B, if A and B then C*

is equivalent to (I) and is irredundant. My postulate sets below have the form (II). But they are free from the irrelevances that (II) usually has when obtained mechanically by Church's rule. The postulates are simple, and in every case the hypothesis is necessary to the conclusion.

There should, of course, be no objection to an irredundant set of postulates simply because it is in form (II).§ For, as Church has shown,¶ *any* irredundant postulate set is equivalent (postulate for postulate) to an irredundant set of form (II). In fact, it can easily be shown that if postulate

* Presented to the Society, San Francisco Section, June 2, 1928.

† Alonzo Church, *On irredundant sets of postulates*, Transactions of this Society, vol. 27 (1925), p. 318.

‡ For the nature of the logic of propositions see B. A. Bernstein, *Sets of postulates for the logic of propositions*, Transactions of this Society, vol. 28 (1926), p. 472. I shall refer to this paper in succeeding footnotes as Paper I.

§ Compare H. M. Gehman, this Bulletin, vol. 32 (1926), p. 159.

¶ Alonzo Church, this Bulletin, vol. 32 (1926), p. 626.

set (I) is irredundant, then set (II) is irredundant and is equivalent to (I), postulate for postulate.* This being so, not only should there be no objection to (II) as a form for irredundant postulates, but (II) should be a *desired normal* form for such postulates. When a set of postulates is in this form, its irredundance can be established by simply exhibiting the necessary independence systems. My postulate sets below are, then, in the normal irredundant form, and they have been so chosen as to be simple and free from the irrelevances usually found in a set blindly made irredundant by Church's mechanical method.

I give three sets of postulates: set A, set B, and set C. The undefined notions in these sets are, respectively,

$$K, \oplus, \odot\,; \quad K, \mid\,; \quad K, \oplus.$$

These undefined notions have been used by me before in non-irredundant sets of postulates for the logic of propositions.† For each set I give the necessary independence systems. Since the postulates are in the normal irredundant form, the independence systems for a set of postulates are also irredundance systems for that set and, together with a consistency system, constitute its "complete existential theory."

2. *Set* A. *Postulates in Terms of Addition and Multiplication.* The set of postulates 1–3 below is the irredundized form of my former set I.‡ The undefined notions for this set are $K, \oplus, \odot$.

1. There is a unique K-element 0 such that, for every K-element a,

$$0 \oplus a = a, \quad 0 \odot a = 0.$$

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* Church showed that the postulates (I) *A, B, C* are equivalent respectively to (III) *A, if A then AB, if A and AB then ABC.* Set (III) has redundancies which (II) has not.

† Paper I. The sets referred to are numbered I, II, III. Set IV of that paper, consisting of but a single postulate, is of course (trivially) irredundant (and "completely independent").

‡ See Paper I.

2. If the element 0 of Postulate 1 exists, there is a unique K-element $1 \neq 0$ such that, for every K-element a,

$$1 \oplus a = 1, \quad 1 \odot a = a.$$

3. If the K-elements 0, 1 of Postulates 1 and 2 exist, they are the only K-elements.

3. *Irredundance of Set* A. Postulate set A is in the normal irredundant form. Hence the following independence systems $(K, \oplus, \odot)$ prove the irredundance of the set. In these systems, the elements 0, 1 are the Boolean *zero* and the *whole* respectively; e' is the logical *negative* of e; $a+b$ and ab are the logical *sum* and logical *product* respectively.

$\overline{1}$. $K : 0,1$; $\quad a \oplus b = 1$; $\quad a \odot b = ab$.

$\overline{2}$. $K : 0,1$; $\quad a \oplus b = ab$; $\quad a \odot b = 1$.

$\overline{3}$. $K : 0,1,e,e'$; $\quad a \oplus b = a + b$; $\quad a \odot b = ab$.

4. *Set* B. *Postulates in Terms of Rejection.* Postulates 1′–3′ following are the postulates of my former set II* recast in an irredundant form. The notions left undefined are $K, \mid \cdot$ ($a \mid b$ may be read: "not-a and not-b").

1′. K consists of two elements.

2′. If K consists of two elements and a is one of the elements, then $a \mid a$ is the other element.

3′. If K consists of two elements, and a is one of the elements and $a \mid a$ the other, then $a \mid (a \mid a)$ and $(a \mid a) \mid a$ are one and the same K-elements.

5. *Irredundance of Set* B. The following independence systems prove the irredundance of set B. The elements and the operations are all Boolean.

$\overline{1'}$. $\quad K : 0$; $\quad a \mid b = 0$.

$\overline{2'}$. $\quad K : 0,1$; $\quad a \mid b = ab$.

$\overline{3'}$. $\quad K : 0,1$; $\quad a \mid b = b'$.

6. *Set* C. *Postulates in Terms of Addition.* Postulates 1″–3″ below are my former set III† made irredundant.

* See Paper I.

† See Paper I.

The undefined notions are K, $\oplus$.

$1''$. There is a unique K-element 0 such that, for every K-element a,

$$0 \oplus a = a.$$

$1''$. If the element 0 of Postulate 1 exists, there is a unique K-element $1 \neq 0$ such that, for every K-element a,

$$1 \oplus a = 1.$$

$3''$. If the K-elements 0, 1 of Postulates 1 and 2 exist, they are the only K-elements.

7. *Irredundance of Set* C. The independence systems proving the irredundance of set C follow. As in the case of sets A and B, the elements and the operations of these systems are all Boolean.

$\overline{1''}$. $\quad K : 0,1 ; \quad a \oplus b = 1.$

$\overline{2''}$. $\quad K : 0,1 ; \quad a \oplus b = a'b.$

$\overline{3''}$. $\quad K : 0,1,e,e' ; \quad a \oplus b = a + b.$

8. *Consistency and Sufficiency of Sets* A, B, C. The following arithmetic systems* prove that sets A, B, C are consistent:

A. $K : 0,1 ; \quad a \oplus b = ab + a + b \bmod 2 ; \quad a \odot b = ab.$

B. $K : 0,1 ; \quad a \mid b \ = ab + a + b + 1 \bmod 2.$

C. $K : 0,1 ; \quad a \oplus b = ab + a + b \bmod 2.$

The sufficiency of each of the sets A, B, C for the logic of propositions is obvious from a comparison of A, B, C with my former sets I, II, III,† respectively.

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* See B. A. Bernstein, *Complete sets of representations of two-element algebras*, this Bulletin, vol. 30 (1924), p. 24.

† See Paper I.

THE DETERMINATION OF PLANE NETS CHARACTERIZED BY CERTAIN PROPERTIES OF THEIR LAPLACE TRANSFORMS*

BY C. T. SULLIVAN

In a previous number of this Bulletin,† Professor J. O. Hassler discusses plane nets whose first and minus first Laplace transforms each degenerate into a straight line, and finds their canonical differential equations. The determination of these equations requires the solution of two partial differential equations of the first order in two dependent and two independent variables. Since two of Hassler's conditions are $H=K=0$, it follows from the well known theory of the equation of Laplace that the second of equations (1) below can be integrated by quadratures. It will be shown that the entire system (1) can be integrated by quadratures, and a fundamental set of integrals for the canonical system will be obtained.

In order to avoid unnecessary repetition of explanations and computations already contained in Hassler's paper, or in works to which he refers, we shall confine our attention entirely to the integration of the differential equations of the problem. These equations have the form

$$(1) \qquad \begin{cases} y_{11} = a^{(11)}y_1 + b^{(11)}y_2 + c^{(11)}y, \\ y_{12} = a^{(12)}y_1 + b^{(12)}y_2 + c^{(12)}y, \\ y_{22} = a^{(22)}y_1 + b^{(22)}y_2 + c^{(22)}y, \end{cases}$$

where $y_1=\partial y/\partial u$, $y_2=\partial y/\partial v$, etc., and where $a^{(ij)}$, $b^{(ij)}$, $c^{(ij)}$, $(i, j=1, 2)$, are functions of u and v, defined in the following manner:

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* Presented to the Society, December 27, 1928.

† This Bulletin, vol. 34 (1928), p. 591.

$$
(2)\quad \begin{cases} a_1^{(22)} + a^{(22)}b^{(11)} = 0, \quad b_2^{(11)} + a^{(22)}b^{(11)} = 0, \\ a^{(11)} = \dfrac{1}{3}(b^{(11)} - 1), \quad c^{(11)} = \dfrac{2}{9}(1 + b^{(11)} + [b^{(11)}]^2) \\ \qquad\qquad - \dfrac{1}{3}(b_1^{(11)} + b_2^{(11)}), \\ a^{(12)} = \dfrac{1}{3}(1 - a^{(22)}), \quad b^{(12)} = -a^{(11)}, \\ c^{(12)} = \dfrac{1}{9}(2a^{(22)}b^{(11)} + a^{(22)} + b^{(11)} - 1), \\ b^{(22)} = -a^{(12)}, \quad c^{(22)} = \dfrac{2}{9}(1 + a^{(22)} + [a^{(22)}]^2) \\ \qquad\qquad - \dfrac{1}{3}(a_1^{(22)} + a_2^{(22)}). \end{cases}
$$

Thus all the functions $a^{(ij)}$, $b^{(ij)}$, $c^{(ij)}$ are expressible in terms of the two $a^{(22)}$, $b^{(11)}$, and their derivatives, while these are defined by the first two equations. The relations defining $a^{(22)}$, $b^{(11)}$ show that

$$b^{(11)}du + a^{(22)}dv = d\psi(u,v),$$

and

$$a^{(22)} = \psi_2, \qquad b^{(11)} = \psi_1, \qquad a_1^{(22)} = b_2^{(11)} = \psi_{12}.$$

Hence we require the solution of

$$\psi_{12} + \psi_1\psi_2 = 0.$$

On integrating with respect to u and v, in turn, we find

$$e^{\psi}\psi_2 = A(v), \quad \psi = \log(\alpha(u) + \beta(v)),$$

where $\alpha(u)$, $\beta(v)$ are arbitrary functions of their respective arguments. The functions $a^{(22)}$, $b^{(11)}$ are therefore* given by

$$(3)\qquad a^{(22)} = \frac{\beta'}{\alpha + \beta}, \quad b^{(11)} = \frac{\alpha'}{\alpha + \beta},$$

the primes indicating differentiation.

* Hassler acknowledges his indebtedness to G. E. Raynor for these solutions, but does not indicate the method by which they were obtained.

The invariants of the second of equations (1) are

$$-H \equiv a_1^{(12)} - a^{(12)}b^{(12)} - c^{(12)} = -\tfrac{1}{3}(a_1^{(22)} + a^{(22)}b^{(11)}) = 0,$$

$$-K \equiv b_2^{(12)} - a^{(12)}b^{(12)} - c^{(12)} = -\tfrac{1}{3}(b_2^{(11)} + a^{(22)}b^{(11)}) = 0.$$

We now transform equations (1) by means of the substitution

$$y = e^{-V}Y,$$

where

$$V = -\int b^{(12)}du + a^{(12)}dv.$$

The system (1) becomes

$$(4) \qquad \begin{cases} Y_{11} + Y_1 - b^{(11)}(Y_1 + Y_2 + Y) = 0, \\ Y_{12} = 0, \\ Y_{22} + Y_2 - a^{(22)}(Y_1 + Y_2 + Y) = 0. \end{cases}$$

From the second of (4) we find

$$Y = G(u) + H(v).$$

Eliminate $Y_1 + Y_2 + Y$ between the first and third of equations (4), and take cognizance of (3); there results the equation

$$\frac{\alpha'}{G'' + G'} = \frac{\beta'}{H'' + H'}.$$

Since u and v are independent variables, we infer that

$$(5) \qquad G'' + G' = k\alpha, \quad H'' + H' = k\beta',$$

where k is a constant. On integrating these equations by the usual method, we find

$$G(u) = ke^{-u}\int e^{u}\alpha du + le^{-u} + l',$$

$$H(v) = ke^{-v}\int e^{v}\beta dv + me^{-v} + m',$$

where l, m, l', m' are additional constants. Hence the most general expression for Y is

$$Y = k\left[e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv\right] + le^{-u} + me^{-v} + l' + m'.$$

In order that the first and third of equations (4) may be satisfied by this value of Y, the constant $l'+m'$ must vanish. The equations (4) have therefore the following three linearly independent integrals:

$$Y^{(1)} = e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv,$$

$$Y^{(2)} = e^{-u}, \quad Y^{(3)} = e^{-v}.$$

If we substitute for $a^{(12)}$ and $b^{(12)}$ in the expression for V, and effect the resulting quadratures, we find

$$V = -\tfrac{1}{3}(u+v) + \tfrac{1}{3}\log(\alpha+\beta).$$

Consequently we have

$$y = \frac{e^{(u+v)/3}}{(\alpha+\beta)^{1/3}}Y.$$

Finally we have the integrals of the canonical differential equations of the most general plane net possessing the property that its first and minus first Laplace transforms each degenerate to a straight line in the form

$$y^{(1)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}\left[e^{-u}\int e^{u}\alpha du + e^{-v}\int e^{v}\beta dv\right],$$

$$y^{(2)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}e^{-u}, \quad y^{(3)} = \left(\frac{e^{u+v}}{\alpha+\beta}\right)^{1/3}e^{-v}.$$

Since α and β are arbitrary, it is possible to write these equations in a form entirely free from quadratures.

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ON CERTAIN LOCI OF LINES INCIDENT WITH CURVES AND SURFACES IN FOUR-SPACE

BY B. C. WONG

In this paper is presented a partial list of formulas giving the orders of the ruled hypersurfaces and ruled surfaces whose rulings are incident with given curves and surfaces in 4-space. If the number of incidences is equivalent to six simple conditions, the number of lines having these incidences is finite. The formulas herein presented are analogous to those giving the orders of the surfaces whose rulings are lines satisfying three conditions and the number of lines satisfying four conditions in 3-space.* Some of these formulas are obvious, others require proof. Those that are not herein included are left out for future consideration.

We shall let h be the number of apparent double points, p the deficiency, of a given curve C^m; H the number of apparent double points, P the deficiency, of a 3-space section c^n of a given surface F^n; and t the number of apparent triple points of F^n or the number of lines through a given point meeting F^n three times.†

First consider the hypersurface V^μ whose lines satisfy four conditions. For a line to meet a given curve once is equivalent to two conditions. Hence the ∞^2 lines cutting across two given curves C^{m_1}, C^{m_2} form a hypersurface whose order is obviously

$$\mu = m_1 m_2. \tag{1}$$

* Salmon, *Analytic Geometry of Three Dimensions*, 5th ed., vol. II, §458–475.

† The formula for t is $\mu\nu(\mu-1)(\mu-2)(\nu-1)(\nu-2)/6$ if F^n is the complete intersection of two hypersurfaces of orders μ and ν respectively. If F^n is only a partial intersection, it is necessary to find t by some other means. See B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35 (1929), pp. 339–343.

If the two curves have s points in common, s must be deducted from the above formula. The order, then, is equal to the number of apparent intersections of the two curves.

The order of the bisecant hypersurface of a given C^m is equal to the number of apparent double points of C^m, that is,

$$\mu = h = \frac{(m-1)(m-2)}{2} - p. \tag{2}$$

The multiplicity of C^m on the hypersurface is, as we shall see, $m-2$.

The V^μ whose ∞^2 lines meet a curve C^m and two surfaces F^{n_1}, F^{n_2} each once is of order

$$\mu = 2mn_1n_2. \tag{3}$$

The proof of this formula is analogous to that of the formula for the order of the surface whose rulings meet three given curves in S_3.*

If C^m meets F^{n_1} and F^{n_2} in s_1 and s_2 points respectively and if F^{n_1} and F^{n_2} meet in a curve of order k, then the quantity $s_1n_2+s_2n_1+km$ has to be deducted from the formula.

The bisecants of an F^n that meet a C^m form a locus of order

$$\mu = m[H + n(n-1)/2] = m[(n-1)^2 - P]. \tag{4}$$

The quantity $s(n-1)$ has to be deducted if C^m and F^n have s points in common.

The ∞^2 lines that are incident with four given surfaces of orders n_1, n_2, n_3, n_4 respectively form a V^μ of order

$$\mu = 3n_1n_2n_3n_4\dagger. \tag{5}$$

Each of the four surfaces is of multiplicity equal to the product of the orders of the other three and each of the points common to two of the surfaces is of multiplicity equal to the sum of their orders multiplied by the product of the orders of the other two surfaces.

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* Salmon, loc. cit., §467.

† For the case $n_1=n_2=n_3=n_4=1$, see Bertini, *Projecktive Geometrie Mehrdimensionaler Räume*, 1924, Chap. 8, §§25–36.

The order of the V^{μ} whose lines meet a given surface F^n twice and two given surfaces F^{n_1}, F^{n_2} each once is

$$(6)\quad \mu = n_1 n_2[2H + n(n-1)/2] = n_1 n_2[(n-1)(3n-4) - 4P]/2.$$

To prove this formula, obtain the order μ' of the hypersurface whose lines are incident with two planes π, π' and with F^n twice. An S_3 through π' meets π in a line l and F^n in a curve c^n. The lines that meet l once and c^n twice form a surface of order $H+n(n-1)/2$. Since l is H-fold on the surface, π is H-fold on $V^{\mu'}$. For the same reason π' is H-fold. Since S_3 meets $V^{\mu'}$ in a composite surface made up of π' counted H-ply and a surface of order $H+n(n-1)/2$, $V^{\mu'}$ is of order

$$\mu' = 2H + n(n-1)/2.$$

Replacing π, π' by F^{n_1}, F^{n_2}, we obtain the formula above.

To obtain the order of the V^{μ} whose lines are trisecants of an F^n and are incident with an F^{n_1}, we note that, t being the number of apparent triple points on F^n, F^{n_1} is t-fold on the hypersurface. Since $(n-2)(6H+n-n^2)/6$ is the order of the trisecant surface* of an S_3-section of F^n, we have the formula

$$(7)\qquad \mu = n_1[t + (n-2)(6H + n - n^2)/6].$$

Now we pass to the scrolls whose rulings are lines satisfying five simple conditions. The order of the scroll of lines meeting a curve C^m once and three surfaces F^{n_1}, F^{n_2}, F^{n_3} each also once is

$$(8)\qquad \nu = 3mn_1n_2n_3.$$

To see this, consider the surface $F^{\nu'}$ whose lines meet a given line l and the three surfaces F^{n_1}, F^{n_2}, F^{n_3}. Any S_3 through l meets F^{n_1}, F^{n_2}, F^{n_3} in three curves c^{n_1}, c^{n_2}, c^{n_3} respectively. The order of the surface in S_3 whose lines meet c^{n_1}, c^{n_2}, c^{n_3} is $2n_1n_2n_3$. Since from each point of l, $n_1n_2n_3$ lines can be drawn incident with F^{n_1}, F^{n_2}, F^{n_3}, l is to be regarded as $n_1n_2n_3$-fold

* Salmon, loc. cit., §471.

on $F^{\nu'}$. Hence $F^{\nu'}$ is of order $\nu' = 3n_1n_2n_3$, for it is met by S_3 in the line l counted $n_1n_2n_3$-ply and the $2n_1n_2n_3$ lines that cut across l, c^{n_1}, c^{n_2}, c^{n_3}. If we replace l by C^m, we have formula (8).

The following formulas can be proved in a similar manner:

$$\nu = mn_1[2H + n(n-1)/2] \tag{9}$$

for the order of the scroll whose lines meet C^m once, F^n twice, and F^{n_1} once;

$$\nu = m[t + (n-2)(6H + n - n^2)/6] \tag{10}$$

for the order of the surface whose lines are trisecants of F^n and unisecants of C^m;

$$\nu = n[h + m(m-1)/2] = n[(m-1)^2 - p] \tag{11}$$

for the order of the surface formed by the bisecants of C^m that are incident with F^n; and

$$\nu = 2m_1m_2n \tag{12}$$

for the order of the surface whose rulings meet C^{m_1}, C^{m_2}, F^n each once.

The orders of the surfaces whose generators meet five surfaces of orders n_1, n_2, n_3, n_4, n_5 each once; one surface F^n twice and three surfaces of orders n_1, n_2, n_3 each once; and one surface F^n three times and two surfaces of orders n_1, n_2 each once are, respectively,

$$\nu = 5n_1n_2n_3n_4n_5, \tag{13}$$

$$\nu = n_1n_2n_3(3H + n^2 - n), \tag{14}$$

$$\nu = n_1n_2[t + (n-2)(6H + n - n^2)/3]. \tag{15}$$

Passing to the formulas for the number of lines that satisfy six simple conditions, we first consider the number N of trisecants to a curve C^m. Let C^m be made up of m lines l_i such that each meets its consecutive line, l_1 and l_m being skew. The number of triples of lines that are all skew is equal to the number of lines which meet a proper C^m, assumed rational, three times. This number is $(m-2)(m-3)(m-4)/6$. If C^m is of deficiency p, the quantity $(m-4)p$ is to be de-

ducted. Hence the formula for the number of trisecants to a curve of order m and deficiency p is

$$N = \frac{(m-2)(m-3)(m-4)}{6} - (m-4)p \cdot \tag{16}$$

The number of lines that meet C^m twice and C^{m_1} once is, from (2),

$$N = m_1 h = m_1\left[\frac{(m-1)(m-2)}{2} - p\right]. \tag{17}$$

We now prove that the multiplicity of C^m on its bisecant hypersurface V^h [see formula (2)] is $m-2$. A line l incident with C^m meets V^h in $h-x$ points distinct from C^m, where x is the multiplicity of C^m. There are $h-x$ lines that meet C^m twice and l once. This number plus the number, $(m-2) \cdot (m-3)(m-4)/6-(m-4)p$, of lines that meet C^m three times is equal to the number of lines that meet a curve C^{m+1} of order $m+1$ and deficiency p three times. Hence

$$h - x + \frac{(m-2)(m-3)(m-4)}{6} - (m-4)p$$
$$= \frac{(m-1)(m-2)(m+3)}{6} - (m-3)p,$$

the right-hand member being the result of changing m to $m+1$ in (16). Replacing h by $[(m-1)(m-2)/2]-p$, we have $x=m-2$, which was to be proved.

The following formulas require no explanation. From (1), (3), (4) we obtain, respectively,

$$N = m_1 m_2 m_3 \tag{18}$$

for the number of lines that meet three curves of orders m_1, m_2, m_3, each once;

$$N = 2m_1 m_2 n_1 n_2 \tag{19}$$

for the number of lines incident with two curves of orders m_1, m_2 and two surfaces of orders n_1, n_2;

$$N = mm_1[(n-1)^2 - P] \tag{20}$$

for the number of lines that meet two curves of orders m, m_1 each once and one surface of order n twice. Formulas (5) or (8), (6) or (9), and (7) or (10) yield, respectively,

$$N = 3mn_1n_2n_3n_4 \tag{21}$$

lines meeting a curve of order m once and four surfaces of orders n_1, n_2, n_3, n_4 each once;

$$N = mn_1n_2[2H + n(n-1)/2] \tag{22}$$

lines meeting C^m, F^{n_1}, F^{n_2} each once and F^n twice; and

$$N = mn_1[t + (n-2)(6H + n - n^2)/6] \tag{23}$$

lines that meet F^n three times, F^{n_1} and C^m each once.

We also have the formulas

$$N = nn_1[(m-1)^2 - p] \tag{24}$$

for the number of bisecants of C^m that meet F^n, F^{n_1} each once;

$$N = 5n_1n_2n_3n_4n_5n_6 \tag{25}$$

for the number of lines incident with six surfaces;

$$N = n_1n_2n_3n_4(3H + n^2 - n) \tag{26}$$

for the number of lines meeting F^n twice and four other surfaces each once; and

$$N = n_1n_2n_3[t + (n-2)(6H + n - n^2)/3] \tag{27}$$

for the number of trisecants of F^n that are incident with three other surfaces.

There are other formulas that call for attention. We omit them for the present.

THE UNIVERSITY OF CALIFORNIA

GROUPS WHICH ADMIT AUTOMORPHISMS IN WHICH EXACTLY THREE-FOURTHS OF THE OPERATORS CORRESPOND TO THEIR INVERSES

BY G. A. MILLER

It is well known that a necessary and sufficient condition that a given group G be abelian is that each of the operators of G correspond to its inverse in one of the possible automorphisms of G, and that in an automorphism of an abelian group all the operators must correspond to their inverses if more than half of them satisfy this condition. Moreover, it is known that in a non-abelian group at most three-fourths of the operators correspond to their inverses in a possible automorphism. The object of the present article is to determine some fundamental properties of the category of groups which separately admit an automorphism in which exactly three-fourths of the operators correspond to their inverses. Such an automorphism will be called a *three-fourths automorphism.*

Suppose that G is a group for which a three-fourths automorphism has been established, and that s_1 and s_2 are any two of its operators which are both non-commutative and correspond to their inverses in this automorphism, while s_α is any operator of G which corresponds to its inverse in the same automorphism. A necessary and sufficient condition that $s_1 s_\alpha$ correspond to its inverse in this automorphism is that s_1 and s_α be commutative. Since $s_1 s_\alpha$ must correspond to its inverse for at least two-thirds of the possible values of s_α it results that s_1 is commutative with exactly one-half of the operators of G, and that these operators constitute an abelian subgroup of G since all of its operators correspond to their inverses in an automorphism. Since an abelian subgroup of index 2 must also correspond

to s_2, and these two subgroups are distinct, it results that G is a non-abelian group which has more than one abelian subgroup of index 2. It therefore contains exactly three such subgroups and has a commutator subgroup of order 2.

In the given three-fourths automorphism of G all of the operators of two of its abelian subgroups of index 2 correspond to their inverses while only half of those of the third such subgroup satisfy the same condition. The remaining operators of this third abelian subgroup obviously correspond in this automorphism to their inverses multiplied by the commutator of order 2. Moreover, the order of G is divisible by 8, and G is the direct product of an abelian group of odd order and a group of order 2^m which admits a three-fourths automorphism. It is obvious that every such direct product admits exactly three three-fourths automorphisms, one for each of its pairs of abelian subgroups of index 2. Hence the following theorem has been established.

THEOREM 1. *If a group admits one three-fourths automorphism it admits exactly three such automorphisms, and it is the direct product of a group of order 2^m which has this property and an abelian group of odd order. Moreover, every such direct product admits a three-fourths automorphism.*

Each of the three three-fourths automorphisms of G is of order 1 or 2, and the three products of these automorphisms in pairs are the three non-identity inner automorphisms of G. As these products are of order 2 the three three-fourths automorphisms of G are commutative and generate a group which is either of order 4 or of order 8. In the former case this group is the group of inner automorphisms of G, and each of the three three-fourths automorphisms of G is an inner automorphism of G, which may be the identity automorphism for one of them, while in the latter case none of the three-fourths automorphisms is an inner automorphism and their continued product is the characteristic outer automorphism of G in which each of the operators of the central of G corresponds to its inverse while each of the remaining operators of G corresponds to its inverse multiplied by the com-

mutator of order 2 contained in G. This continued product is obviously also a characteristic automorphism when the three-fourths automorphisms are inner automorphisms of G.

A necessary and sufficient condition that one of the three three-fourths automorphisms of G be the identity automorphism is that exactly one-fourth of the operators of G have orders which exceed 2. Hence this condition characterizes the groups which are direct products of the octic group and an abelian group of order 2^m and of type $(1, 1, 1, \cdots)$.* The continued product of the three three-fourths automorphisms in this case is the characteristic inner automorphism of order 2 in which all the operators of the subgroup of index 2 generated by the operators of order 4 contained in G correspond to themselves. The other two three-fourths automorphisms in this case are the two non-characteristic inner automorphisms of G. In the other case where the three three-fourths automorphisms of G generate a group of order 4, they must therefore all be inner automorphisms of order 2 and hence G must be the Hamiltonian group of order 2^m. That is, G must be the direct product of the quaternion group and the abelian group of order 2^{m-3} and of type $(1, 1, 1, \cdots)$.

From the preceding paragraph it results that when the three three-fourths automorphisms of G generate a group of order 4 then G is the direct product of an abelian group of order 2^α and of type $(1, 1, 1, \cdots)$, and either the octic group or the quaternion group. When G is the direct product of any abelian group which is not of the given type and one of the two non-abelian groups of order 8 its three three-fourths automorphisms will generate a group of order 8 and of type $(1, 1, 1, \cdots)$, and when the three three-fourths automorphisms of G generate such a group this will also be the case of the direct product of G and any abelian group whatever. While every group which admits three-fourths automorphisms is the direct product of its Sylow subgroups,

* G. A. Miller, *Annals of Mathematics*, vol. 7 (1906), p. 55.

and only one of these Sylow subgroups is non-abelian, this non-abelian subgroup H of order 2^m may itself be the direct product of an abelian group and a non-abelian group. This can obviously not be the case when it has only two independent generators. We shall prove that it is always the case when it has more than three such generators and that it may or may not be the case when it has exactly three such generators.

Let H_0 be the central of H and suppose the operators of H are arranged in co-sets with respect to H_0. We select s_1, s_2 from two distinct co-sets so that their orders are as low as possible, and when their orders are different we assume that the order of s_1 is the larger. If H_0 contains any operator whose order is equal to that of s_1, we select all the operators whose orders are at least equal to this order in any given reduced set of independent generators of H_0. The group generated by these operators has only the identity in common with the group generated by s_1, since s_1 is an operator of lowest order in the co-set to which it belongs. We therefore adjoin s_1 to the operators already selected in the given set of independent generators of H. These operators when s_1 is replaced by s_1^2 appear in a reduced set of independent generators of H_0 if the order of s_1 exceeds 2. If the order of s_1 is 2 the order of s_2 is also 2 and the non-identity commutator of H is an operator of order 2 in the group generated by the selected set of independent generators of H_0. If this operator is not a power of one of the operators of this set they may evidently be re-selected in such a way that it is such a power, and H is the direct product of the group generated by s_1, s_2 and this operator, and the group generated by the rest of the operators in the given set of independent generators of H_0. In what follows we shall therefore assume that the order of s_1 exceeds 2 and hence s_1^2 is an operator in a reduced set of independent generators of H_0.

When the order of s_2 is equal to that of s_1 we may add also s_2^2 to these generators and augment the set thus obtained by other operators in a reduced set of independent generators of H_0. The only case when this cannot be done is when $s_1^2 = s_2^2$

and hence the order of s_1 is 4, but this order may also be 4 when $s_1^2 \neq s_2^2$. In the former case s_1, s_2 generated the quaternion group while they generate a group of order 16 or of order 32 in the latter case. When the order of s_2 is less than the order of s_1 we adjoin to the operators of the given set of reduced independent generators of H_0 others, if any exist, whose orders are equal to or greater than that of s_2. We thus obtain a set of reduced independent generators of H_0 when the order of s_2 is 2. When this order exceeds 2 we add s_2^2 to this set and complete a set of reduced independent generators of H_0 in the usual way. Hence we may state the following theorem.

THEOREM 2. *If H is any non-abelian group of order 2^m which involves more than one abelian subgroup of index 2, then it is possible to select all the operators of a set of independent generators of H in such a way that only one pair of them is composed of non-commutative operators and that the commutator of order 2 is a power of one of these independent generators.*

From this theorem it follows that when the commutator of order 2 in H is a power of s_1 or of s_2, then H is the direct product of the group generated by s_1, s_2, and the group generated by all the remaining operators of the given set of independent generators of H. In all other cases we adjoin to s_1, s_2 the independent generators whose power is the commutator of order 2, and H is then the direct product of the group generated by these three operators and the group generated by the remaining operators in the given set of independent generators of H. This proves the following theorem.

THEOREM 3. *Whenever H has more than three independent generators it is the direct product of an abelian group and a non-abelian group which has at most three independent generators of H for its independent generators.*

When it has three such generators they may be so selected that one of them appears in the central of H.

Whenever a non-abelian group G involves three abelian subgroups of index 2 then either all of these subgroups are

conjugate under the group of isomorphisms of G or two of them are thus conjugate while the third is a characteristic subgroup of G. Hence it results that the group of inner isomorphisms of such a group involves at most one characteristic operator of order 2 under its group of isomorphisms. When the three three-fourths automorphisms of G generate a group of order 8 it therefore results that this group may involve either only one or three operators of order 2 which are characteristic under the group of isomorphisms of G. If it involves only one such operator its other six operators of order 2 are composed of two sets of three conjugate operators under this group of isomorphisms. In the other possible case the four non-characteristic operators of order 2 appear in two sets of two conjugates under the same group.

It was noted above that when the three three-fourths automorphisms of G generate a group of order 4, then the central of G can involve only operators of order 2 besides the identity. If these automorphisms generate a group of order 8 and the central of G is generated by operators of order 2 then the three abelian subgroups of index 2 under G are either all of type $(2, 1, 1, \cdots)$, or two of them are of this type while the third is of type $(1, 1, 1, \cdots)$. In the latter case an operator t which transforms G according to a three-fourths automorphism and has its square in G gives rise to a group which has an abelian subgroup of index 2. This is also obviously the case when t transforms G according to an operator in its group of inner isomorphisms but it cannot happen when the central of G involves operators whose orders exceed 2. This gives rise to the following theorem.

THEOREM 4. *There is one and only one group G of order 2^m, $m>4$, which satisfies the two conditions that its central involves no operator whose order exceeds 2 and that it appears in a group of order 2^{m+1} involving operators which transform exactly three-fourths of the operators of G into their inverses but involving no abelian subgroup of index 2.*

Since the Hamiltonian groups of order 2^m constitute such a well known and important category of groups it may be of interest to note here that it results from the preceding developments that these groups are characterized by the facts that they admit three-fourths automorphisms and that the continued product of these three automorphisms is the identity. This category is also characterized by the fact that it is composed of the groups whose three three-fourths automorphisms are inner automorphisms of order 2. The entire category of Hamiltonian groups may be similarly characterized by the fact that it is composed of the groups whose centrals involve no operators of order 4 and which admit such three three-fourths automorphisms that in their continued product all the operators of the central correspond to their inverses and at least one operator in each central co-set corresponds to itself.

THE UNIVERSITY OF ILLINOIS

THE ROLE OF DIRECTION IN THE VARIATIONS OF A DEFINITE INTEGRAL IN SPACE

BY LOUIS BRAND

1. *Introduction.* In the space problem of the calculus of variations, whether treated in the parametric form as by Bliss and Mason,† or in terms of angle as by Rider‡ or Sakellariou,§ the Euler equations are three in number but connected by a relation. Thus but two of the equations are independent. The object of this paper is to show that two independent Euler equations arise from variations parallel

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* Presented to the Society, April 16, 1927.

† Mason, M. and Bliss, G. A., *The properties of curves in space which minimize a definite integral*, Transactions of this Society, vol. 9 (1908), pp. 440–466.

‡ Rider, P. R., *The space problem of the calculus of variations in terms of angle*, American Journal of Mathematics, vol. 39 (1917), pp. 241–256.

§ Sakellariou, N., *The space problem of the calculus of variations*, Palermo Rendiconti, vol. 44 (1920), pp. 53–68.

to principal normal and binormal, respectively, of the curve giving an extremum. Tangential variations yield nothing but an identity in partial differentiation. The two independent Euler equations may be replaced by a vector equation which involves a certain vector $\boldsymbol{Q}$. This vector plays an important part in the entire theory covering the first and second variations of the integral.

2. *The Euler Equations.* Consider the integral

$$I = \int_0^l F(\boldsymbol{r}, \boldsymbol{t})ds$$

taken along a curve Γ in space joining two fixed points. Here F is a scalar point function of the position vector $\boldsymbol{r}$ and its derivative $d\boldsymbol{r}/ds = \boldsymbol{t}$ (the unit tangent vector) which has continuous partial gradients of the second order with respect to $\boldsymbol{r}$ and $\boldsymbol{t}$. We note in passing that this formulation of the problem applies equally well to variations in space or on a given surface.

Now let the curve Γ, $\boldsymbol{r} = \boldsymbol{r}(s)$, give I an extreme value. We replace Γ by another curve Γ_1:

$$\boldsymbol{r}_1 = \boldsymbol{r} + \epsilon(u\boldsymbol{t} + v\boldsymbol{p} + w\boldsymbol{b}) \tag{1}$$

between the given points, where u, v, w are three independent functions of s of class C' which vanish for $s = 0$ and $s = l$, and $\boldsymbol{t}, \boldsymbol{p}, \boldsymbol{b}$ are unit vectors in the direction of the tangent, principal normal and binormal along Γ, chosen so that $\boldsymbol{tpb}$ form a right-handed set. Thus (1) gives explicit representation to variations in the directions of $\boldsymbol{t}, \boldsymbol{p}$ and $\boldsymbol{b}$.

The value of I along Γ_1 is

$$I(\epsilon) = \int_0^l F(\boldsymbol{r}_1, \boldsymbol{t}_1)\phi(s, \epsilon)ds,$$

where

$$\phi(s, \epsilon) = ds_1/ds,$$

and if $\nabla_r F$, $\nabla_t F$ denote partial gradients of F with respect to $\boldsymbol{r}$ and $\boldsymbol{t}$,

$$(2)\quad I'(\epsilon) = \int_0^l \left\{\left[\frac{\partial \boldsymbol{r}_1}{\partial \epsilon}\cdot\nabla_{\boldsymbol{r}_1}F + \frac{\partial \boldsymbol{t}_1}{\partial \epsilon}\cdot\nabla_{\boldsymbol{t}_1}F\right]\phi + F\frac{\partial \phi}{\partial \epsilon}\right\}ds.$$

On differentiating (1) with respect to ϵ and with respect to s and making use of Frenet's formulas:

$$\frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{p}, \qquad \frac{d\boldsymbol{p}}{ds} = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}, \qquad \frac{d\boldsymbol{b}}{ds} = -\tau\boldsymbol{p},$$

we obtain

$$(3)\qquad \frac{\partial \boldsymbol{r}_1}{\partial \epsilon} = u\boldsymbol{t} + v\boldsymbol{p} + w\boldsymbol{b},$$

$$(4)\quad \boldsymbol{t}_1\phi(s,\epsilon) = (1 + \epsilon u' - \epsilon\kappa v)\boldsymbol{t} + \epsilon(\kappa u + v' - \tau w)\boldsymbol{p} + \epsilon(\tau v + w')\boldsymbol{b}$$

where the primes denote differentiation with respect to s. We now differentiate (4) with respect to ϵ:

$$(5)\quad \frac{\partial \boldsymbol{t}_1}{\partial \epsilon}\phi + \boldsymbol{t}_1\frac{\partial \phi}{\partial \epsilon} = (u' - \kappa v)\boldsymbol{t} + (\kappa u + v' - \tau w)\boldsymbol{p} + (\tau v + w')\boldsymbol{b},$$

whence

$$(6)\quad \frac{\partial \phi}{\partial \epsilon} = (u' - \kappa v)\boldsymbol{t}\cdot\boldsymbol{t}_1 + (\kappa u + v' - \tau w)\boldsymbol{p}\cdot\boldsymbol{t}_1 + (\tau v + w')\boldsymbol{b}\cdot\boldsymbol{t}_1.$$

Putting $\epsilon = 0$ in (4), (6) and (5) in turn now gives

$$\phi(s,0) = 1, \qquad \phi_\epsilon(s,0) = u' - \kappa v,$$

$$\left.\frac{\partial \boldsymbol{t}_1}{\partial \epsilon}\right|_{\epsilon=0} = (\kappa u + v' - \tau w)\boldsymbol{p} + (\tau v + w')\boldsymbol{b} = (u\boldsymbol{t})' + (v\boldsymbol{p})' + (w\boldsymbol{b})' + (\kappa v - u')\boldsymbol{t}.$$

Finally we note that for a given $\boldsymbol{r}$, $\boldsymbol{t}$ can only vary over a unit sphere; hence $\boldsymbol{t}\cdot\nabla_{\boldsymbol{t}}F = 0$. With these results and (3), (2) becomes

$$I'(0) = \int_0^l \{(u\boldsymbol{t} + v\boldsymbol{p} + w\boldsymbol{b})\cdot\nabla_{\boldsymbol{r}}F + [(u\boldsymbol{t})' + (v\boldsymbol{p})' + (w\boldsymbol{b})']\cdot\nabla_{\boldsymbol{t}}F + (u' - \kappa v)F\}ds,$$

and by the usual integration by parts

$$I'(0) = \int_0^l \left\{ u\left(\boldsymbol{t}\cdot\nabla_r F - \boldsymbol{t}\cdot\frac{d}{ds}\nabla_t F - \frac{dF}{ds}\right) + v\left(\boldsymbol{p}\cdot\nabla_r F - \boldsymbol{p}\cdot\frac{d}{ds}\nabla_t F - \kappa F\right) + w\left(\boldsymbol{b}\cdot\nabla_r F - \boldsymbol{b}\cdot\frac{d}{ds}\nabla_t F\right)\right\} ds.$$

Putting $v=w=0$, $w=u=0$, $u=v=0$ in turn, the condition $I'(0)=0$ yields the three Euler equations:

$$\boldsymbol{t}\cdot\nabla_r F - \boldsymbol{t}\cdot\frac{d}{ds}\nabla_t F - \frac{dF}{ds} = 0, \tag{7}$$

$$\boldsymbol{p}\cdot\nabla_r F - \boldsymbol{p}\cdot\frac{d}{ds}\nabla_t F - \kappa F = 0, \tag{8}$$

$$\boldsymbol{b}\cdot\nabla_r F - \boldsymbol{b}\cdot\frac{d}{ds}\nabla_t F = 0. \tag{9}$$

3. *The Euler Equation for the Tangential Variation is an Identity.* To show this we differentiate the relation $\boldsymbol{t}\cdot\nabla_t F=0$ with respect to s:

$$\frac{d\boldsymbol{t}}{ds}\cdot\nabla_t F + \boldsymbol{t}\cdot\frac{d}{ds}\nabla_t F = 0.$$

Thus (7) reduces to an identity in partial differentiation,

$$\frac{dF}{ds} = \frac{d\boldsymbol{r}}{ds}\cdot\nabla_r F + \frac{d\boldsymbol{t}}{ds}\cdot\nabla_t F.$$

The two independent Euler equations thus arise from variations parallel to the principal normal and binormal.

4. *The Euler Equation in Vector Form.* Equations (7), (8), (9) are the $\boldsymbol{t}$, $\boldsymbol{p}$ and $\boldsymbol{b}$ components of the vector equation

$$\nabla_r F - \frac{d\boldsymbol{Q}}{ds} = 0, \quad \text{where} \quad \boldsymbol{Q} = F\boldsymbol{t} + \Delta_t F. \tag{10}$$

This equation puts in evidence three cases of immediate integrability, namely when $\nabla_r F$ is zero, parallel to a fixed vector $\boldsymbol{k}$, or perpendicular to $\boldsymbol{k}$; in these cases $\boldsymbol{Q}$, $\boldsymbol{k}\times\boldsymbol{Q}$

or $\boldsymbol{k}\cdot\boldsymbol{Q}$ are respectively constant. Equation (10) is of the second order when $\nabla_t\boldsymbol{Q}\neq 0$; it degenerates when $\boldsymbol{Q}$ is a function of $\boldsymbol{r}$ alone, that is, only when F has the form $\boldsymbol{t}\cdot f(\boldsymbol{r})$.

In a paper presented to the Society,† I have shown that (10) follows directly from the condition $I'(0)=0$ by using variations of an arbitrary but fixed direction, $\boldsymbol{r}_1=\boldsymbol{r}+\epsilon v\mathbf{c}$, where $\mathbf{c}$ is a constant vector. The second variation then takes the form $I''(0)=\mathbf{c}\cdot\Phi\cdot\mathbf{c}$, where Φ is a tensor of the second order.

To show the part played by the vector $\boldsymbol{Q}$ in the general theory we merely state the following results. The Weierstrass-Erdmann corner condition is

$$\boldsymbol{Q}(\boldsymbol{r},\boldsymbol{t}^+)=\boldsymbol{Q}(\boldsymbol{r},\boldsymbol{t}^-).$$

In a *regular* problem the tensor $\nabla_t\boldsymbol{Q}$, which is planar and symmetric, is not zero; its second scalar invariant (in the sense of Gibbs) must be positive or zero in the case of a minimum. This is the analogue of the Legendre condition.

If $\boldsymbol{t}=\boldsymbol{t}^*$ in a field of extremals and $\boldsymbol{Q}^*=\boldsymbol{Q}(\boldsymbol{r},\boldsymbol{t}^*)$, the Euler equation gives

$$\boldsymbol{t}^*\times\operatorname{rot}\boldsymbol{Q}^*=0$$

in this field. The Hilbert invariant integral is

$$I^*=\int\boldsymbol{Q}^*\cdot\boldsymbol{t}ds.$$

From this we have the fundamental theorem of Weierstrass

$$\nabla I=\int(\boldsymbol{Q}-\boldsymbol{Q}^*)\cdot\boldsymbol{t}ds.$$

The integrand $(\boldsymbol{Q}-\boldsymbol{Q}^*)\cdot\boldsymbol{t}$ is the E-function. The transversality condition is simply $\boldsymbol{Q}^*\cdot\boldsymbol{t}=0$.

In the corresponding problem on a surface of unit normal $\boldsymbol{n}$, the Euler equation is

† December 1, 1928, *The calculus of variations from a vector standpoint.* See this Bulletin, vol., 35 (1929), p. 153.

$$\boldsymbol{n} \times \left(\nabla_r F - \frac{d\boldsymbol{Q}}{ds} \right) = 0,$$

when $\boldsymbol{Q}$ is defined as above. The tensor $\nabla_t \boldsymbol{Q}$ is now symmetric and linear and its first scalar invariant must be positive or zero in the case of a minimum. In a field of extremals

$$\boldsymbol{n} \cdot \operatorname{rot} \boldsymbol{Q}^* = 0.$$

The Hilbert integral, the E-function and the transversality condition are the same as before.† The last equation and $\boldsymbol{Q}^* \cdot \boldsymbol{n} = 0$ show that $\boldsymbol{Q}^*$ is the surface gradient of a scalar function W. The curves $W = \text{const.}$ are the transversals of the field.

We note, finally, that analogous results hold in the problem of finding an extremum for a surface integral

$$I = \int F(\boldsymbol{r}, \boldsymbol{n}) d\sigma$$

bounded by a fixed closed curve. The Euler equation in this case is

$$\boldsymbol{n} \cdot \nabla_r F + \operatorname{div} \boldsymbol{P} = 0 \quad \text{where} \quad \boldsymbol{P} = F\boldsymbol{n} + \nabla_n F$$

and div denotes a *surface* divergence. That $\boldsymbol{P}$ has the same general form as $\boldsymbol{Q}$ is apparent. If $\boldsymbol{n} = \boldsymbol{n}^*$ in a field of extremal surfaces and $\boldsymbol{P}^* = \boldsymbol{P}(\boldsymbol{r}, \boldsymbol{n}^*)$, then

$$\operatorname{div} \boldsymbol{P}^* = 0.$$

The Hilbert integral, the E-function and the transversality condition are respectively

$$\int \boldsymbol{P}^* \cdot \boldsymbol{n} d\sigma, \quad (\boldsymbol{P} - \boldsymbol{P}^*) \cdot \boldsymbol{n}, \quad \boldsymbol{P}^* \cdot \boldsymbol{n} = 0.$$

These results will be developed elsewhere.

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† From this fact we have an immediate proof of a theorem given by Douglas in this Bulletin (vol. 32 (1926), pp. 669–674) to the effect that transversality of an integral relative to a surface Σ is the section by Σ of its space transversality.

EFFECTS OF THE OSCNODE UPON COVARIANT FORMS OF THE RATIONAL PLANE QUARTIC CURVE*

BY J. H. NEELLEY

1. *Introduction.* A recent paper† discusses some effects of the tacnode and ramphoid cusp upon the combinants of two line sections and other associated forms of the rational plane quartic curve. This paper observes those forms for the curve with an oscnode. Well known results have been purposely omitted.

2. *The Ordinary Oscnode.* As previously shown‡ the oscnodal parameters may be taken as $t = \pm a$ if the two tangents of the pencil of lines on the oscnode have contacts at $t=0$ and $t=\infty$. These tangents and the join of their contacts give the curve the representation

$$(1) \qquad \begin{cases} x_0 = t^4 - a^2t^2, \\ x_1 = t^2 - a^2, \\ x_2 = 4t^3 - 2\sigma t^2 - 4a^2t, \end{cases}$$

where $\sigma = t_1 - a^2t_1^{-1}$, the sum of the parameters of the points of contact of the distinct double line. Two line sections $(\xi x)=0$, $(\eta x)=0$ have combinants which are converted into covariant curves of R_2^4 by means of the relations

$$\begin{vmatrix} (a\xi) & (b\xi) \\ (a\eta) & (b\eta) \end{vmatrix} = | \, a \; b \; x \, | \, ,$$

and so forth, where the third-order determinants are formed from equations (1).§ The discriminant of the invariant

* Presented to the Society, September 6, 1928.

† Neelley, *Concerning covariants of the rational plane quartic curve with compound singularities*, this Bulletin, vol. 35, pp. 67–72.

‡ Neelley, *Compound singularities* etc., American Journal of Mathematics, vol. 49, p. 397.

§ Rowe, Transactions of this Society, vol. 12, p. 296.

$\overline{12}^4$ (1, 1) of the pencil $(\xi x)+\lambda(\eta x)=0$ is the quadratic form which becomes the conic g_2,* the envelope of the self-apolar line sections of the curve. The condition that the quartics cut out by (ξx) and (ηx) be derived functions of a quintic is another second-order combinant which becomes the Stahl conic K,† the conic on the vertices of the flex-triangles of the first osculants of R_2^4. The pencil $g_2+\lambda k=0$ is used to study the effects of the singularity with only well known results for this type of the curve.

The fundamental involution is determined by the quartics

$$(2)\quad (\beta t)^4 \equiv 4t^3 + 4a^2t, \quad (\gamma t)^4 \equiv t^4 + 2\sigma t + 6a^2t^2 + a^4.$$

The complete system of covariants of these two quartics has the g_3 of the first quartic and $\overline{12}^4(1, H_2)$ and $\overline{12}^4(2, H_1)$ identically zero. We also note that the form $\overline{12}^2(1, 2)$ is a quartic with the same roots as those of the quartic cut from the curve by the reference line $x_0=0$. We shall observe this form for the special cases of the singularity.

3. *The Oscnodal Cusp.* This type of the oscnode assumes the properties of the ramphoid cusp as well as those resulting from the coincidence of three adjacent nodes. If the nodal parameter is $t=\infty$ and the contact of the distinct (improper) double line on the node is $t=0$, the nodal line, this double line, and a line on $t=0$ can be taken so as to represent the curve by‡

$$(3)\quad \begin{cases} x_0 = e, \\ x_1 = 6ct^2, \\ x_2 = 12at^4 + 12dt, \end{cases}$$

with

$$(4)\qquad (\beta t)^4 \equiv 4at, \qquad (\gamma t)^4 \equiv 4at^3 + d.$$

* Rowe, loc. cit., pp. 298–299; Salmon, *Modern Higher Algebra*, 4th edition, p. 219.

† Stahl, Journal für Mathematik, vol. 101, pp. 300–325.

‡ Winger, American Journal of Mathematics, vol. 36, p. 65.

Consideration of the combinants of (ξx) and (ηx) gives the conics g_2 and K as

$$aex_1^2 + c^2x_0x_2 = 0 \tag{5}$$

and

$$cx_0x_2 = 0 \tag{6}$$

respectively. So K reduces to two sides of the triangle of reference, the oscnodal line being one. Obviously the pencil of conics $g_2+\lambda K$ is a double-contact pencil and the conics of the pencil have contact with R_2^4 at the cusp. The conic on the flexes and the points q_i* is of this pencil and has the equation

$$3\,aex_1^2 - c^2x_0x_2 = 0. \tag{7}$$

As this is on the point (1, 0, 0) and as it has previously been shown that three flexes and one point q coincide at a ramphoid cusp† we have the following theorem.

THEOREM 1. *The oscnodal rational plane quartic curve with a cusp has its points q_i on the distinct double line.*

But more interesting is the fact that the general equation of the Stahl conic N‡ reduces to the square of this distinct double line. Now the conic N may be defined as the locus of a point from which a pencil of lines will cut a pencil of quartics from the curve R_2^4 for which the invariant I_2' vanishes. The line obviously satisfies such a condition as any line $(\lambda x)=0$ and $x_1=0$ give such a pencil of quartics. We have therefore the following result.

THEOREM 2. *The conic N meets the curve in two pairs of coincident points, the points q_i, when the oscnode is a cusp.*

The covariant system of the fundamental quartics (4) has several forms which vanish identically. The quartics

* Rowe, Transactions of this Society, vol. 13, p. 390.

† Neelley, *Compound singularities* etc., American Journal of Mathematic, vol. 49, p. 390.

‡ Stahl, Journal für Mathematik, vol. 101, pp. 300–325.

are self-apolar and each is apolar to the Hessian of the other. This is characteristic of the oscnodal cusp. The ramphoid cusp covariant, the quadratic resulting from operating with the first of the quartics upon the Jacobian of the second and its Hessian,* vanishes here also. Then the quartic $\overline{12}^2(1, 2)$ is the same as that cut out by the line $x_1=0$. So again this form is a line section of the curve. The covariant forms of this system are very degenerate and many syzygetic relations exist.

4. *The Knot.* This special form of the oscnode is such that any line on the singular point meets the curve in three coincident points and the oscnodal line meets it in four, all with the same parameter value. The oscnodal line at $t=0$ and the line on the oscnode and the point $t=\infty$ with the tangent at the latter point give R_2^4 as follows:

$$(8) \qquad \begin{cases} x_0 = at^4, \\ x_1 = 4bt^3, \\ x_2 = 6ct^2 + 4dt + e, \end{cases}$$

with

$$(9) \qquad (\beta t)^4 \equiv dt^4 + et^3, \qquad (\gamma t)^4 \equiv ct^3 + dt^2.$$

The conic g_2 has the equation

$$(10) \qquad (4bdx_0 + aex_1)^2 = 0,$$

which is the square of a line on the oscnode. The conic K is the square of the oscnodal line

$$(11) \qquad x_0^2 = 0.$$

So the pencil $g_2+\lambda K$ is a degenerate one with its vertex at the oscnode. The conic K has all of its intersections with R_2^4 at $t=0$ and since six of those of g_2 and the curve are at the same point the conic on the flexes has at least six intersections with R_2^4 at the oscnode. Now the points q_i are given by $\overline{12}^3(1, 2)$ of the system of the quartics (9) and this form

* Neelley, *Concerning covariants* etc., this Bulletin, vol. 35, p. 72.

is the quadratic $t^2=0$. Hence we have the following theorem.

THEOREM 3. *The two points q_i coincide at the knot on the oscnodal rational plane quartic curve.*

The resultant of the line sections is the very simple equation of the curve given below:

$$(12)\quad a^3ex_1^4 + 16a^2bdx_0x_1^3 + 96ab^2cx_0^2x_1^2 - 256b^4x_0^3x_2 = 0.$$

Again the forms of the system of the fundamental quartics are in many cases dependent. Many relations vanish identically; all of the independent quadratic forms do except $\overline{12}^3(1, 2)$ and $\overline{12}^3(2, H_1)$. Once more we find that $\overline{12}^2(1, 2)$ is apolar to each of the fundamental quartics and is consequently a line section. When formed for types of the curve without an oscnode $\overline{12}^2(1, 2)$ is not a line section, but it has been shown to be for all types with the oscnode, so we suggest the following theorem.

THEOREM 4. *The quartic form $\overline{12}^2(1, 2)$ of the system of the fundamental quartics of the rational plane quartic curve locates four points on a line if the curve has an oscnode, the line being on the oscnode.*

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TOWNSEND ON REAL VARIABLES

Functions of Real Variables. By E. J. Townsend. New York, Henry Holt, 1928. 405 pp. $5.00.

This book, as indicated in the preface, is a text-book on the subject, and not a treatise. The material used has been divided into seven chapters of approximately equal length, except for the first. They are as follows: Chapter I, *Real Number System*,—a short discussion of real numbers as defined by Cantor and Dedekind; Chapter II, *Theory of Point Sets*,—the usual properties involved in the notions of open and closed sets and measure, chiefly confined to linear sets; Chapter III, *Continuity and Discontinuity of Functions*,—including double and iterated limits, continuity in one or two variables, semi-continuous and pointwise discontinuous functions; Chapter IV, *Derivatives and their Properties*,—derivatives, upper and lower derivatives, law of the mean, partial derivatives, and total differentiability; Chapter V, *Riemann Theory of Integration*,—the usual theory, including improper integrals and integrals involving a parameter; Chapter VI, *Lebesgue and Other Integrals*,—the usual definitions and theory, including primitive functions and a short account of the work of Stieltjes' Hellinger, Denjoy, and Perron; Chapter VII, *Infinite Series*,—uniform and quasi-uniform convergence of series of functions, with applications to differentiation and integration, properties of power series (not including Taylor's series), condensation of singularities, and various methods of handling divergent series. Mention should also be made of the collections of problems at the ends of the chapters and the numerous references to original sources, both of which should be useful to teacher and student.

In criticizing this book it would be natural to compare it with the other two books on the same subject in English, those by Hobson and by Pierpont, but it must be remembered that they are essentially treatises, while the book under consideration is a comparatively short text-book and must be judged accordingly. A discussion of its merits falls naturally under three heads: its aesthetic qualities, the choice of material, and its utility.

With regard to the first the book is excellent. Not only is the author's style good, but he has a manner of presentation that makes the book intensely interesting. The publishers, also, are to be congratulated on their workmanship, as the arrangement and printing are artistic and there is a noticeable freedom from misprints.

The choice of material appeals to the writer in general, but not entirely so. The chief sin of omission is the confining of the point-set theory to linear sets and the restriction of so many of the theorems on functions to functions of one variable. To be sure, most of the generalization is immediate, but in some cases this is not true and the beginner cannot be expected to know when this is the case. It also seems strange to see no discussion of Taylor's series and the conditions for its validity in a chapter devoted to infinite series, especially when nearly eight pages are given to power series. On the other hand it is the writer's opinion that the author has to some extent

yielded to the temptation to make his book encyclopedic in character and would have done better to omit certain topics so as to permit a thorough treatment of the topics mentioned above.

One who expresses an opinion as to the utility of this book will be largely influenced by what he conceives to be the primary aim of a first course in this subject, whether the imparting of a large body of new information or training in rigorous mathematical reasoning. The writer regards this book as superior in regard to the first of these objectives, but weak in regard to the second. The chief trouble is a willingness to allow the student to accept as true without proof facts which appear to be obvious, a failing most difficult to overcome in the average graduate student. For instance, in the definition of interior measure on p. 65 there is no mention of the necessity of proving uniqueness. The casual statements in various places (for example, p. 277) that the work can be readily extended to n dimensions is an encouragement to loose thinking. The paucity of references to previous sections in demonstrations, although improving the style, adds to the difficulty in regard to this matter, since the student is apt to accept the statements made without verifying them. Furthermore, the author himself is not impeccable in some of his statements. We learn on p. 44 that the sum of any number of closed sets is a closed set, which is of course not true unless a finite number is meant. But on pp. 51, 52 we find a set E which is defined as the sum of an enumerable system of perfect sets E_n and read that "since E_n is a perfect set for each value of n, it follows that E is also a perfect set." It is regrettable that a book otherwise so fine should be so marred by this sort of thing.

W. A. WILSON

SECOND EDITION OF NETTO'S KOMBINATORIK

Lehrbuch der Kombinatorik. By Eugen Netto. Second edition, with notes and additional chapters by Viggo Brun and Th. Skolem. Leipzig and Berlin, B. G. Teubner, 1927. viii+341 pp.

This work is volume 7 of the series B. G. Teubners Sammlung von Lehrbüchern aus dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, in which a number of contributors to the Encyclopædie expand their articles in textbook form.

The first edition appeared in 1901 and contained thirteen chapters, which are reprinted without change, pp. 1–258. Th. Skolem contributes notes on these chapters (pp. 309–338) containing simplifications of a number of proofs, as well as extensions of several of the topics treated by Netto. Since no review of the first edition appeared in this Bulletin, it seems appropriate to indicate briefly the contents of the first thirteen chapters. Chapters 1 and 2 give the elementary notions on permutations and combinations, and their connection with the binomial and multinomial theorems. Chapter 3 deals with arrangements of a number of elements with restrictions on the places which the elements may occupy; topics such as the problem of eight queens and Latin squares are treated here. Chapter 4 considers the inversions and sequences in permutations, and Chapter 5

gives an elementary exposition of various partition problems. The analytic treatment of partitions by means of generating functions follows in Chapter 6, and Chapter 7 contains some applications of partitions to problems in analysis. Chapter 8 considers the combinatorial arrangements as sums of symbolic products, and Chapter 9 is concerned with some special arrangements other than permutations and combinations. Chapter 10 deals with triads, and although Netto quotes some early work of E. H. Moore, Skolem's notes on this chapter contain no reference to the researches of Cole, White and their students. In Chapter 11, the Kirkman problem is considered, and Chapter 12 gives applications of combinatorial analysis to probability and questions in formal algebra such as the inversion of a power series. Chapter 13 contains a collection of formulas, mainly identical relations between binomial coefficients. The reviewer has found this collection very useful on a number of occasions. The new subject matter of the second edition begins with Chapter 14 (by Viggo Brun). Here a very general combinatorial concept is introduced, that of the distribution function

$$D(s_0 \cdots s,\ r_0 \cdots r,\ \cdots,\ w_0 \cdots w;\ \sigma_0 \cdots \sigma,\ \rho_0 \cdots \rho,\ \omega_0 \cdots \omega)$$

which is defined as the number of distributions of from s_0 to s black balls, from r_0 to r red balls, $\cdots$, from w_0 to w white balls in a number of urns of which the first is to contain from σ_0 to σ balls, the second from ρ_0 to ρ balls, $\cdots$, and the last, from ω_0 to ω balls. First, a duality theorem is noted, which says that in D the sets of Latin and Greek letters may be interchanged without changing the value of the distribution function. The recurrent formulas and explicit expressions for D are given, and the results are applied to an extension of the work of Möbius on the factorization of integers treated in Chapter 8 (where the Möbius number-theoretic function $\mu(n)$ is introduced). Reference is also made to Brun's own work on the sieve of Eratosthenes. In Chapter 15, Th. Skolem deals with groupments, combinatorial reciprocities, and systems of pairs. A groupment is a distribution in the sense of the preceding chapter with the restriction that no urn shall contain two balls of the same color. The notion of duality is introduced and generalized, and it is shown that the groupments of systems of pairs have important connections with the theory of triads, as well as with the Petersen graphs in analysis situs. Lack of space prevents the reviewer from giving any details of these somewhat complicated things, but it should be stated that both these last chapters form important contributions to combinatorial analysis, and that they may be read without knowledge of the preceding chapters.

T. H. GRONWALL

SHORTER NOTICES

The Bakhshâlî Manuscript—A Study in Mediaeval Mathematics. By G. R. Kaye. With photographic facsimiles of the manuscript and transliteration of the text in roman script. Calcutta, Government of India, Central Publication Branch, 1927. Pp. i+156+v. Plates 1+4+42. Price Rs 28 or 43s. 6d.

Bakhshâlî is the name of a village near the city of Peshawar in the north-west corner of India. There in 1881, a farmer unearthed a manuscript of a work on mathematics. The greater portion of it has been destroyed by rough handling and the remains consist of 70 leaves of birch-bark but some of these are mere scraps. That manuscript is now generally known as the Bakhshâlî Manuscript. It is preserved in the Bodleian Library, Oxford. The work of analysing and editing the Manuscript was begun by the distinguished Indologist, Dr. R. Hoernle, but on account of his untimely death, it came to the hands of Mr. G. R. Kaye who has successfully finished it. The editor has added an elaborate and masterly introduction to the text covering 84 pages in print which contains not only an analysis of the contents of the manuscript but also a study of its relation with other Hindu treatises of mathematics as far as possible.

The Bakhshâlî work is a compendium of rules and illustrative examples. It gives also the solutions of the most of the examples together with their verification. It contains material relating to arithmetic, algebra, and geometry (including mensuration). The topics for discussion are found to include fractions, square-root, arithmetical and geometrical progressions, income and expenditure, profit and loss, computation of gold, interest, rule of three, summation of certain complex series, simple equations, simultaneous linear equations, quadratic equations, indeterminate equations of the second degree of a particular type, mensuration and miscellaneous problems. The treatment of all these subjects is commonly included in other Hindu treatises of mathematics. But we miss in it any reference to the treatment of the indeterminate equation of the first degree and the so-called Pellian equation, both of which enter largely into later Hindu works and in the solution of which the Hindus long anticipated the works of Euler and Lagrange.

The most notable mathematical principles in the Bakhshâlî work are the approximate square-root formula,

$$(a^2+r)^{1/2} = a + \frac{r}{2a} - \frac{(r/(2a)^2)}{2(a+r/(2a))},$$

the calculation of the errors of successive order

$$\epsilon_1 = (r/(2a))^2,\ \epsilon_2 = \left(\frac{(r/(2a)^2)}{2(a+r/(2a))}\right)^2$$

and a consequent process of reconciliation. The fundamental arithmetical

operations are indicated by putting after, occasionally before, the quantity affected, the abbreviations of the Sanskrit words of respective import. In the case of subtraction, the abbreviation is found to have degenerated into a simple cross-sign (+). The Bakhshâlî work is particularly characterised by the absence of any kind of algebraic symbolism. This has necessitated the preservation of every detail of the working of the solutions of algebraic problems keeping up their generality throughout so that the final statement of the results should clearly present the whole formulas involved. "Indeed the numerical quantities in those problems are treated almost like algebraic symbols," rightly observes Mr. Kaye. Most of those problems have been solved by the method of *false position*. The place of an unknown quantity in a statement of a problem is indicated by a heavy dot (•) and is designated as *śûnyasthâna* or "empty space," meaning thereby that the number to be placed there has not been ascertained. Other matters of interest are: the least common multiple, the solution of the quadratic equation, the exclusive use of the decimal place-value notation with the cipher, the average value and the change ratios of measures.

The time of the Bakhshâlî work is rather uncertain. On paleographical evidence Mr. Kaye holds that it should be referred to a period about the twelfth century. There are clear signs to show that the present manuscript is an imperfect copy of an older manuscript which again is of the character of a perpetual commentary on a still earlier work. The editor has failed to detect it. But it did not escape the keen eyes of Dr. Hoernle. He is of opinion that the present copy was made about the ninth century but the original work must be referred to the early centuries of the Christian era. There are many things in the work to point to the same period. Mr. Kaye has drawn attention to a few instances of foreign influence in the Bakhshâlî work including the one about the approximate square-root stated above which he says he has not found in any other Hindu works. That formula, correct up to the second approximation, was known in the early Jaina canonical works (about 300 B.C. or still earlier) and also reappears in later works including that of Âryabhata (499 A.D.) as has been pointed out by Rodet. Another has been overstated under a similar misapprehension and the rest are minor and of doubtful value. The editor himself does not wish to press upon the latter.

The significance and importance of a publication like the present one are apparent to all lovers of the history of science. And their thanks are certainly due to Mr. Kaye for the great amount of pains that he has taken in explaining and editing the Bakhshâlî manuscript. There are, no doubt, several inaccuracies in his study of the work, some of which are not negligible. It is not possible to point them out in a short review like this. A critical examination of them will be found in a paper by the present reviewer entitled *The Bakhshâlî mathematics* in the Bulletin of the Calcutta Mathematical Society (vol. 21, pp. 1–60). But they do not lessen the labor of the editor. Moreover such inaccuracies are not unusual in case of all pioneer studies of difficult ancient manuscripts.

BIBHUTIBHUSAN DATTA

Vorlesungen über Algebra. Fourth revised edition of the work with the same title by the late Dr. Gustav Bauer. By L. Bieberbach. Leipzig and Berlin, B. G. Teubner, 1928. x+334 pp.

The well known *Algebra* of Bauer* now appears in a fourth edition remodeled and modernized to such an extent as to form practically a new book. In Section 1, on the fundamental properties of algebraic equations, we may note the thoroughly modern chapter on complex numbers, taken largely from volume 1 of Bieberbach's *Funktionentheorie*, and in Chapter 3, Cauchy's function-theoretic proof of the fundamental theorem of algebra is followed very appropriately by Rouché's theorem, which is used to prove the continuity of the roots as functions of the coefficients. Section 2 deals with determinants, matrices, linear equations, quadratic and bilinear forms, and the proofs are made short and elegant by an extensive use of the vector notation. Section 3 deals with symmetric functions and Tschirnhaus' and Jerrard's transformations in the customary manner. Section 4, which carries the too narrow title "numerical solution of equations," is the most distinctive feature of the book. The first chapter gives various methods for finding upper and lower bounds for the real roots, and then proceeds to Lill's graphical method, which is the most rapid means of obtaining fairly good approximations to the real roots; better approximations may then be obtained by the standard methods which are briefly described. It is pleasing to see less than half a page devoted to Horner's method instead of the inordinate amount of space given to it in American texts. Chapter 2 gives Descartes' rule of signs, and the theorems of Rolle, Budan Fourier, and Sturm, including Hermite's treatment of Sturm's problem by quadratic forms. Chapter 3 contains the method of Cohn (Mathematische Zeitschrift, vol. 14) for determining the number of complex roots inside a circle, theorems of Schur and Kakeya, and Schur's discussion of equations with all the roots having negative real parts.

Chapter 4 gives a thorough discussion of Graeffe's method both for real and complex roots. Chapter 5 begins with the classical theorem of Gauss on the relative position in the complex plane of the zeros of $f(x)$ and $f'(x)$, proceeds to theorems of Poulain, Laguerre, Fejér, Grace, Heywood and Walsh, and also gives composition theorems due to Malo and Schur. The reviewer is particularly pleased to see the very readable account in Chapters 3 and 5 of a number of quite modern and interesting things, which are sadly neglected in most other texts.

Section 5, on the algebraic solution of equations, gives a very readable account, mostly along classical lines, of the elements of cyclotomy, substitution groups and the Galois theory; there is a new proof of the irreducibility of the cyclotomic equation due to Späth (Mathematische Zeitschrift, vol. 26). The book closes with an elementary account of continued fractions.

T. H. GRONWALL

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* Reviewed in this Bulletin: first edition, vol. 10 (1903–4), p. 257, by L. E. Dickson, and second edition, vol. 17 (1911), p. 428, by A. Dresden.

NOTES

The April, 1929, number of the Transactions of this Society (volume 31, No. 2) contains the following papers: *A note on closest approximation*, by D. Jackson; *Homogeneous polynomials with a multiplication theorem*, by O. C. Hazlett; *Generalized factorial series*, by T. Fort; *On extending a continuous* (1-1) *correspondence* (*second paper*, H. M. Gehman; *A determination of all normal division algebras in sixteen units*, by A. A. Albert; *Expansions in generalized Appell polynomials, and a class of related linear functional equations*, by I. M. Sheffer; *Systems of infinitely many linear differential equations of infinite order, with constant coefficients*, by I. M. Sheffer; *On general topology and the relation of the properties of the class of all continuous functions to the properties of space*, by E. W. Chittenden; *Properties of functions represented by the Dirichlet series* $\sum (av+b)^{-s}$, *or by linear combinations of such series*, by J. I. Hutchinson; *Concerning zero-dimensional sets in euclidean space*, by R. L. Wilder; *On the linear partial q-difference equation of general type*, by C. R. Adams; *Les fonctions polygènes comme intégrales d'équations différentielles*, by G. Calugaréano.

The April, 1929, number of the American Journal of Mathematics (volume 51, No. 2) contains: *Singular points of vector fields under general boundary conditions*, by M. Morse; *A complete solution of Laplace's equation by an infinite hypervariable*, by P. W. Ketchum; *Functions with assigned initial values*, by W. J. Trjitzinsky; *The abstract identity of modular systems and ideals*, by D. Harkin; *Concerning the cut points of a continuous curve when the arc curve, AB, contains exactly N independent arcs*, by N. E. Rutt; *A direct treatment of systems of linear differential equations whose coefficients have uniform singularities*, by J. A. Nyswander; *On related difference and differential sytems*, by W. M. Whyburn; *On the motion of a rigid body about a fixed point in space of constant curvature*, by J. Pierpont; *On the prime ideals of the general cubic Galois field*, by C. G. Latimer; *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, by O. Zariski.

The opening number of volume 30, series 2, of the Annals of Mathematics (December, 1928) contains: *On cubic fields*, by A. Arwin; *The impossibility of a separation of types of linear odd divisors of binary quadratic forms*, by R. G. Archibald; *A parametric problem of the calculus of variations and its treatment as a problem of Lagrange*, by L. H. McFarlan; *On the fundamental existence theorems for differential systems*, by W. M. Whyburn; *On a class of Taylor's series*, by P. L. Srivastava; *Quartic equations with certain groups*, by R. Garver; *Canonical forms of plane cubic curves under euclidean transformations*, by R. S. Burington and H. K. Holt; *On forms which repeat under multiplication*, by C. G. Latimer; *On the multiple solutions of the Pell equation*, by D. H. Lehmer; *On certain hitherto unsolved cases of the complex multiplication of elliptic functions*, by S. C. Mitra; *On the complex roots of a*

transcendental equation occurring in the electron theory, by J. Pierpont; *Extended results in elimination*, by T. W. Moore; *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, by P. Alexandroff; *Semi-parallel transformations of lines of curvature*, by M. M. Slotnick.

The new Accademia d'Italia consists of the following classes each with fifteen members: Exact and Natural Sciences, Moral Sciences, Arts, Literature. The first thirty members have recently been selected by the government. The representatives of pure and applied mathematics and physics are F. Severi, professor of calculus in the University of Rome, E. Fermi, professor of theoretical physics in the University of Rome, and G. Vallauri, professor of electrical engineering in the Technical School of Turin.

A Congress of Mathematicians of the Slavic Countries will be held at Warsaw, September 23–27, 1929, under the auspices of the head of the Polish government. Professor W. Sierpinski, of the University of Warsaw, is president of the executive committee of the Congress.

Cambridge University has awarded Smith's prizes to H. D. Ursell, of Clare College, and J. M. Whittaker, of Trinity College. Rayleigh prizes have been awarded to J. Hargreaves, of Clare College, J. G. Semple, of St. John's College, and S. Verblunsky, of Magdalene College.

The Italian Society of Sciences has awarded its prize in mathematics for the year 1928 to Professor C. Rosati, of the University of Pisa, for his researches in algebraic geometry.

The Watson medal of the National Academy of Sciences has been awarded to Dr. Willem de Sitter, director of the observatory at Leiden.

The Research Corporation of New York has awarded its research medal to Professor Werner Heisenberg, of the University of Leipzig, for "brilliant scientific achievement."

Professor Erwin Schrödinger has been elected a member of the Prussian Academy of Sciences.

Professor Heinrich Tietze, of the University of Munich, has been elected a member of the Bavarian Academy of Sciences.

Professor Guido Castelnuovo has been elected correspondent of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor Luigi Bianchi.

The following have been elected correspondents of the Academy of Sciences of Turin: Professors L. Tonelli, G. Vitali, A. Signorini, E. Pistolesi, F. Lori, and E. Fermi.

In the presence of the King of Norway, the University of Oslo commemorated, on March 6, 1929, the one hundredth anniversary of the death of

the Norwegian mathematician Abel. On this occasion honorary degrees were conferred on Professors Jacques Hadamard, G. H. Hardy and Paul Painlevé.

Professors Max von Laue, of the University of Berlin, and Arnold Sommerfeld, of the University of Munich, have been elected honorary members of the Leningrad Academy of Sciences.

Among the recently elected fellows of the Royal Society of Edinburgh are the following mathematicians: Dr. F. Bath, lecturer at the University of St. Andrews, and Mr. R. A. Robb, lecturer at the University of Glasgow.

Professor Pieter Zeeman, of the University of Amsterdam, has been elected an honorary member of the Physical Society of London.

The following have been recently elected foreign associates of the National Academy of Sciences: Dr. Willem de Sitter, director of the observatory at Leiden, Professor Arnold Sommerfeld, of the University of Munich, and Professor C. de la Vallée Poussin, of the University of Louvain.

The King of Italy has conferred the decoration of Officer of the Crown of Italy upon Dr. Raymond Pearl, director of the Institute of Biological Research of Johns Hopkins University.

Dr. Paul Dienes, lecturer in mathematics at University College, Swansea, and formerly of the University of Budapest, has been appointed university reader at Birkbeck College.

Dr. D. R. Hartree, of the Cavendish Laboratory, Cambridge, has been appointed to the Beyer chair of applied mathematics at the University of Manchester.

Professor Friedrich Hund, of the University of Rostock, has been appointed professor of mathematical physics at the University of Leipzig.

Dr. Pascual Jordan has been promoted to an associate professorship of theoretical physics at the University of Hamburg.

Associate Professor Theodor Kaluza, of the University of Königsberg, has been appointed professor of mathematics at the University of Kiel.

Dr. Harold R. Phalen, professor of mathematics and instructor in physics at St. Stephen's College, Columbia University, has been appointed dean of the college.

Dr. Tibor Rado has been appointed Lecturer in Mathematics at Harvard for the first term of the year 1929–1930. He will give a course on "Minimal Surfaces and the Problem of Plateau" together with a course on "The Elementary Theory of Differential Equations."

The University of Lwów has conferred an honorary doctorate on Professor W. Sierpínski, of the University of Warsaw.

Professor Leonida Tonelli has been transferred from the professorship

of algebraic analysis at the University of Bologna to the professorship of infinitesimal analysis at the same University.

Professor C. E. Weatherburn, of Canterbury College, Christchurch, New Zealand, has been appointed professor of mathematics at the University of Western Australia.

The following have been appointed docents: Dr. W. Nikliborc and Dr. J. P. Schander, at the University of Lwów; Dr. Zarankiewicz, at the University of Warsaw.

Professor E. R. Hedrick was selected to give the Fifth Annual Faculty Research Lecture at the University of California at Los Angeles. This lecture was given April 26, 1929, on the subject *Logical reasoning in mathematics and in science.*

Professor H. E. Slaught, of the University of Chicago, delivered lectures during March 1929 on *How mankind learned to count* at Haverford College, Lafayette College, Lehigh University, Rutgers University, and Swarthmore College. These lectures were usually sponsored by mathematics clubs. The lectures were given for the purpose of stimulating interest in mathematics and they were well attended.

The original manuscript of Einstein's *Zur einheitlichen Feldtheorie,* recently published in the Sitzungsberichte of the Prussian Academy of Sciences, has been purchased for Wesleyan University by Mr. G. W. Davison and Mr. A. W. Johnston, two of the University's trustees.

A symposium on theoretical physics will be conducted at the University of Michigan, from June 24 to August 16, 1929. The following courses of lectures will be delivered: by E. A. Milne, of the University of Oxford, Problems in astrophysics, and vector and tensor methods in statics and dynamics; by K. F. Herzfeld, of Johns Hopkins University, Statistical mechanics; by Léon Brioullin, of the University of Paris, Quantum statistics; by Edward Condon, of Princeton University, Introduction to quantum mechanics; by P. A. M. Dirac, of the University of Cambridge, Advanced quantum mechanics; by D. M. Dennison, of the University of Michigan, Band spectra.

Assistant Professor H. M. Gehman, of Yale University, has been appointed head of the department of mathematics at the University of Buffalo.

Mr. W. J. Hazard has been promoted to an assistant professorship of mathematics at the University of Colorado.

Assistant Professors Lester S. Hill and Evelyn Walker have been promoted to associate professorships of mathematics at Hunter College.

Miss Marie Litzinger has been promoted to an assistant professorship of mathematics at Mount Holyoke College.

Professor W. V. Lovitt, of Colorado College, has been appointed Dean of Men; he retains his professorship of mathematics.

Assistant Professor A. D. Michal, of the Ohio State University, has been appointed associate professor of mathematics at the California Institute of Technology.

Professor W. E. Milne, who has been on leave as professor of mathematics at Stanford University this year, will return to the University of Oregon next fall. During his absence Dr. H. C. Hicks has served as assistant professor of mathematics at the University of Oregon.

Associate Professor L. L. Smail, of Lehigh University, has been promoted to a professorship of mathematics.

Dr. P. A. Smith has been promoted to an assistant professorship of mathematics at Barnard College, Columbia University.

Dr. W. J. Trjitzinsky has been promoted to an assistant professorship of mathematics at Lehigh University.

Professor James V. Uspensky of the Russian Academy of Sciences has been appointed professorial lecturer at the University of Minnesota for the summer quarter, and part of the spring quarter. During the spring quarter he lectured on *Integration in finite form*. He will lecture on the *Theory of numbers* in the first session of the summer quarter and on the *Theory of probability* in the second session.

Mr. Charles H. Vehse, of Brown University, has been appointed assistant professor of mathematics at the University of West Virginia.

Dr. Louis Weisner has been promoted to an assistant professorship of mathematics at Hunter College.

The following appointments to instructorships are announced:

Barnard College, Columbia University: Dr. Lulu Hofmann, Mr. H. W. Raudenbush.

Brown University, Mr. G. N. Carmichael, Mr. K. G. Fuller of Northwestern University.

Harvard University, Dr. T. F. Cope.

Hunter College, Mr. Aubrey W. Landers, Jr.

The Johns Hopkins University, Dr. Oscar Zariski (Associate).

Swarthmore College, Mr. Aubrey H. Smith.

University of Wyoming, Mr. Richard W. Warner, Mr. C. N. Shuster.

Jorge Duclout, engineer and honorary professor of the theory of elasticity at the University of Buenos Aires, died February 15, 1929, at the age of seventy-five.

Professor Erik Schou, of the Copenhagen Technical School, died November 8, 1928, at the age of fifty-five.

NEW PUBLICATIONS

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BROWN (F. G. W.). Progressive trigonometry. Part 1. London, Macmillan, 1928. 10+222 pp.

CARR (H. W.). Leibniz. London, Benn, 1929. 6+222 pp.

CATTANEO (P.). Appunti sui numeri reali. Padova, Zannoni, 1928. 64 pp.

CISOTTI (U.). Lezioni di calcolo tensoriale. Milano, Libreria Editrice Politecnica, 1928. 7+94 pp.

CORRAL (J. I.). Cantidades complejas. (Memorias Cientificas de la Direccion de Montes y Minas, Cuba, tomo 1.) Habana, 1929. 160 pp.

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DICKSON (L. E.). Höhere Algebra. Autorisierte Deutsche Ausgabe von L. E. Dickson "Modern algebraic theories" herausgegeben von E. Bodewig. Leipzig, Teubner, 1929. 8+242 pp.

DINTZL (E.) und VASELLI (C.). Aufgaben aus der reinen und angewandten Mathematik. Band 2. Wien, Hölder-Pichler-Tempsky, 1928. 107 pp.

DUNNE (J. W.). An experiment with time. 2d edition, revised. London, A. and C. Black, 1929. 8+213 pp.

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ESTÉVE (R.) et MITAULT (H.). Algèbre et trigonométrie. 2 volumes. Paris, Gauthier-Villars, 1928. 552+314 pp.

FANTAPPIÈ (L.) See VOLTERRA (V.).

FETTWEIS (E.). Methodik für den Rechenunterricht. Paderborn, Schöningh, 1929. 272 pp.

FLADT (K.). Elementarmathematik. Band I, Elementargeometrie, Teil 2. Leipzig, Teubner, 1928. 8+181 pp.

FORSYTH (A. R.). A treatise on differential equations. 6th edition London, Macmillan, 1929. 18+583 pp.

FOWLER (R. H.). The elementary differential geometry of plane curves. 2d edition. Cambridge, University Press, 1929. 9+105 pp.

FRY (T. C.). Elementary differential equations. New York, Van Nostrand, 1929. 10+255 pp.

FULCO (P.). Congruenze algebriche ed esponenziali. Applicazioni. Memoria IIa. Civitavecchia, 1929. 230 pp.

GARABEDIAN (C. A.) and WINSTON (J.). Plane trigonometry. New York, McGraw-Hill, 1929. 10+306 pp.

GRUNDEL (F.). Die Mathematik an den deutschen höheren Schulen. Teil 2. Leipzig, Teubner, 1929. 6+148 pp.

HAAG (J.). Cours complet de mathématiques élémentaires. Tome IV: Trigonométrie. Paris, Gauthier-Villars, 1929.

HAUPT (O.). Einführung in die Algebra. 2 Bände. Leipzig, Akademische Verlagsgesellschaft, 1929. 367+15+663+13 pp.

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

BOULDER, COLORADO, SUMMER MEETING AND COLLOQUIUM, August 27-30, 1929.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than July 30. Professor Virgil Snyder will deliver his retiring presidential address, entitled *The problem of the cubic variety in four-way space*; and Professor R. L. Moore will deliver a series of colloquium lectures on *Point-set theory*.

NEW YORK CITY, October 26, 1929.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than October 5.

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 29, 1929.

Abstracts must be in the hands of Associate Secretary B. A. Bernstein, University of California, Berkeley, Calif., not later than November 1. By invitation of the Program Committee, Professor M. W. Haskell will deliver an address on *Autopolar configurations in two and three dimensions*.

ANN ARBOR, MICHIGAN, November 29-30, 1929.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 7.

BETHLEHEM, PENNSYLVANIA, ANNUAL MEETING, December 27-28, 1929.

Abstracts must be in the hands of Associate Secretary M. H. Infessor R. G. D. Richardson, 501 West 116th St., New York City, not later than November 30. A symposium on the differential equations of applied mathematics will be arranged, and will be announced in detail later.

DES MOINES, IOWA, December 30-31, 1929.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 30. There will be the usual joint session with Section A of the American Association for the Advancement of Science and with the Mathematical Association of America. The Josiah Willard Gibbs Lecture will be held on Monday afternoon. More detailed announcements will be made later.

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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